Kolmogorov's Theorem for Low-Dimensional Invariant Tori of Hamiltonian Systems

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Abstract

In this paper the problem of persistence of invariant tori under small perturbations of integrable Hamiltonian systems is considered. The existence of one-to-one correspondence between hyperbolic invariant tori and critical points of the function Ψ of two variables defined on semi-cylinder is established. It is proved that if unperturbed Hamiltonian has a saddle point, then under arbitrary perturbations there persists at least one hyperbolic torus.

Key words: invariant tori; KAM theory; Percival's variational principle.
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1   Notation

First, we introduce some notation which is used troughout of the paper. Every vector $\mathbf{v}$ in m-dimensional Euclidean space is assumed to be a column

vector. Row vectors are denoted by $\mathbf{v}^\top$. Hence $\mathbf{v} = (v_1, \ldots, v_m)^\top$. For a matrix $\mathbf{A} = (A_{ij})$, $i, j = 1, \ldots, n$, with entries A_{ij}, i is the row index and j the column index, and $\mathbf{A}^\top = (A_{ji})$ stands for the transposed matrix. The product of a matrix $\mathbf{A}$ by a column vector $\mathbf{v}$ is denoted by $\mathbf{A}\mathbf{v}$. It is the column vector with the components $(\mathbf{A}\mathbf{v})_i = A_{ij}v_j$. The product of a row vector $\mathbf{v}^\top$ and a matrix $\mathbf{A}$ is the row vector with the components $(\mathbf{v}^\top\mathbf{A})_j = v_i A_{ij}$. We thus get $\mathbf{v}^\top\mathbf{A} = (\mathbf{A}^\top\mathbf{v})^\top$. The product of two matrices $\mathbf{C} = \mathbf{A}\mathbf{B}$ is a matrix with the entries $C_{ij} = A_{ik}B_{kj}$ with the summation convention over the repeated indices. The scalar product of two vectors is $\mathbf{v}^\top \cdot \mathbf{u} = v_i u_i$. By abuse of notation, we also denote by $\mathbf{u} \cdot \mathbf{v} = u_i v_i$ the scalar product of two column vectors.

The tensor product of two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ is the matrix $A := \mathbf{u} \otimes \mathbf{v}$ with the entries $A_{ij} := u_i v_j$, $i, j = 1, \ldots, n$. For the product of this matrix with a vector we have

$$(\mathbf{u} \otimes \mathbf{v})\mathbf{w} = (\mathbf{v}^\top \cdot \mathbf{w})\mathbf{u} \quad \text{and} \quad \mathbf{w}^\top(\mathbf{u} \otimes \mathbf{v}) = (\mathbf{w}^\top \cdot \mathbf{u})\mathbf{v}.$$

There is a difference between the Jacobian $\mathbf{v}'(x)$ of a vector function $\mathbf{v} : \mathbb{R}^n \to \mathbb{R}^m$ and its gradient: the Jacobian is denoted by

$$\mathbf{v}' := (\partial_{x_j} v_i) = \left[\frac{\partial \mathbf{v}}{\partial x_1}, \frac{\partial \mathbf{v}}{\partial x_2}, \ldots, \frac{\partial \mathbf{v}}{\partial x_n} \right],$$

and the gradient is its transpose

$$\nabla\mathbf{v} = {\mathbf{v}'}^\top = (\partial_{x_i} v_j) = [\nabla v_1, \nabla v_2, \nabla v_3]$$

If a function $\mathbf{v} : \mathbb{R}^{n-k} \times \mathbb{R}^k$ depends on two variables $y \in \mathbb{R}^{n-k}$ and $z \in \mathbb{R}^k$, then

$$\mathbf{v}'_y(y, z) = (\partial_{y_j} v_i)_{ij} = \left[\frac{\partial \mathbf{v}}{\partial y_1}, \frac{\partial \mathbf{v}}{\partial y_2}, \ldots, \frac{\partial \mathbf{v}}{\partial y_{n-k}} \right],$$

and $\nabla_y \mathbf{v} = (\mathbf{v}'_y)^\top$. For the derivatives of a scalar function H we will use the notation

$$\frac{\partial H}{\partial y}(y, z) = H'_y(y, z) \text{ and } \frac{\partial^2 H}{\partial z \partial y}(y, z) = \nabla_z(H'_y)(y, z). \qquad (1.1)$$

In other words, $\partial^2 H/\partial z \partial y$ is the matrix with the entries $\partial^2 H/\partial z_i \partial y_j$ and

$$\frac{\partial^2 H}{\partial z \partial y} = \left(\frac{\partial^2 H}{\partial y \partial z} \right)^\top.$$

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Let $\varrho > 0$. Denote by Σ_ϱ the complex neighborhood

$$\Re_\varrho = \{\boldsymbol{\xi} \in \mathbb{T}^{n-1} + i\mathbb{R}^{n-1} : \operatorname{Re}\xi \in \mathbb{T}^{n-1},\ |\operatorname{Im}\xi_j| \le \varrho,\ \ 1 \le j \le n-1\} \quad (1.2)$$

of the (n-1)-dimensional torus $\mathbb{T}^{n-1}$. We will identify $\Re_\varrho$ with the strip

$$\{\boldsymbol{\xi} : \operatorname{Re}\xi \in \mathbb{R}^{n-1},\ |\operatorname{Im}\xi_j| \le \varrho,\ \ 1 \le j \le n-1\} \subset \mathbb{C}^{n-1}.$$

We denote by B_ϱ^n the complex neighborhood

$$B_\varrho^n = \{\boldsymbol{p} \in \mathbb{C}^n : |\operatorname{Re}\boldsymbol{p}| \le \varrho \in D,\ |\operatorname{Im}p_j| \le \varrho,\ \ 1 \le j \le n\} \subset \mathbb{C}^n. \quad (1.3)$$

.

2 Problem formulation. Results

Invariant tori. In this paper we consider the problem of persistence of quasi-periodic motions spanning lower dimensional tori in a nearly-integrable Hamiltonian system

$$\dot{\boldsymbol{p}} = -\partial_{\boldsymbol{q}} H(\boldsymbol{p}, \boldsymbol{q}), \quad \dot{\boldsymbol{q}} = \partial_{\boldsymbol{p}} H(\boldsymbol{p}, \boldsymbol{q}) \tag{2.1}$$

with the Hamiltonian

$$H(\boldsymbol{p}, \boldsymbol{q}) = H_0(\boldsymbol{p}) + \varepsilon H_1(\boldsymbol{p}, \boldsymbol{q}). \tag{2.2}$$

Here the functions H_0 and H_1 are analytic on the set $D \times \mathbb{T}^n$, where $D \subset \mathbb{R}^n$ is a neighborhood of zero and $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$ is n-dimensional torus. In particular, H_1 is an analytic 2π-periodic function of the angle variable $\boldsymbol{q} = (q_1, \ldots, q_n)$. For $\varepsilon = 0$ the system is integrable and the phase space is foliated by invariant n-dimensional invariant tori $\{\mathbf{q} = \overline{\boldsymbol{\omega}}\, t + \text{const.}, t \in \mathbb{R}\}$ with the frequency vectors $\overline{\boldsymbol{\omega}} = \nabla H_0(\boldsymbol{p})$, $\boldsymbol{p} \in D$. If all components of $\overline{\boldsymbol{\omega}}$ are rationally independent, then KAM theory, see [11] and [1], shows that, under suitable non-degeneracy assumptions, such invariant tori persist under small analytic perturbations.

An invariant n-dimensional invariant torus of the unperturbed system is said to be resonance if the number of rationally independent components of $\overline{\boldsymbol{\omega}}$ is $m < n$. A resonance invariant torus of the unperturbed system is foliated by invariant m-dimensional tori. A resonance torus breaks-up under

small perturbation and only a few of its constituent invariant m-dimensional tori survive a perturbation.

The mathematical study of low-dimensional tori in Hamiltonian systems dates back to the late 1960s. The intensive treatment of the problem starts with pioneering papers [14] and [19]. We refer reader to the monographs [2], [4], [12], [15], [20] and the papers [9], [10], [21], and [24] for the state of art in the domain.

In this paper we deal with the problem of persistence of (n-1)- dimensional tori in setting close to the original Kolmogorov theorem. Our goal is to find conditions on the unperturbed Hamiltonian H_0 under which system (2.1) has an invariant torus of dimension $n-1$ for *every analytic perturbation and all sufficiently small ε*. This problem is still poor investigated. We refer to papers [5] and [8] for results and discussion.

We focus on the problem on persistence of hyperbolic invariant tori for arbitrary analytic perturbation of a Hamiltonian function. Throughout of the paper we assume that the frequency vector $\overline{\boldsymbol{\omega}} = \nabla_p H(0,0)$ has $n-1$ rationally independent components and the Hessian $H_0''(0,0)$ is nondegenerate. Then there exists, see [21], an affine symplectic transformation with rational coefficients such that in new variables

$$\boldsymbol{p} = (\boldsymbol{y}, z_2) \in \mathbb{R}^{n-1} \times \mathbb{R}, \quad \boldsymbol{q} = (\boldsymbol{x}, z_1) \in \mathbb{T}^{n-1} \times \mathbb{T}, \quad \boldsymbol{z} = (z_1, z_2),$$

the Hamiltonian $H(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = H_0(\boldsymbol{y}, z_1) + H_1(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ satisfies the following condition.

H.1 Recall (1.2) and (1.3). There is $\varrho > 0$ with the properties. The function $H_0(\boldsymbol{y}, z_2)$ is analytic in the complex ball $(\mathbf{y}, z_2) \in B_{3\varrho}^n$; the perturbation $H_1(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ is analytic on the cartesian product of the complex strip $(\mathbf{x}, z_1) \in \Re_{3\varrho}$ and the ball $(\mathbf{y}, z_2) \in B_{3\varrho}^n$. In particular, H_1 is 2π-periodic in $\boldsymbol{x}$ and z_1. Moreover, there is $c > 0$ such that

$$\sup_{(\boldsymbol{y}, z_2) \in B_{3\rho}^n} |H_0(\boldsymbol{y}, z_2)| + \sup_{(\boldsymbol{y}, z_2) \in B_{3\rho}^n, \ (\boldsymbol{x}, z_1) \in \Re_{3\rho}} |H_1(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})| \leq c, \qquad (2.3)$$

H.2 The frequency vector $\overline{\boldsymbol{\omega}} = (\boldsymbol{\omega}, 0)$ and Hamiltonian H_0 satisfy

$$\nabla_{\boldsymbol{y}} H_0(0,0) = \boldsymbol{\omega} \in \mathbb{R}^{n-1}, \quad \partial_{z_2} H_0(0,0) = 0. \qquad (2.4)$$

The components of $\boldsymbol{\omega}$ are rationally independent. and satisfy the diophantine condition

$$\left| (\boldsymbol{\omega}^\top \cdot \boldsymbol{s})^{-1} \right| \leq c_0 |\boldsymbol{s}|^{-n} \quad \text{for all} \quad \boldsymbol{s} \in \mathbb{Z}^{n-1} \setminus \{0\}. \qquad (2.5)$$

In the variables $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ system (2.1) reads

$$\dot{\boldsymbol{x}} = \nabla_{\boldsymbol{y}} H, \quad \dot{\boldsymbol{y}} = -\nabla_{\boldsymbol{x}} H, \quad \dot{\boldsymbol{z}} = \mathbf{J}\nabla_{\boldsymbol{z}} H, \quad \text{where} \quad \mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (2.6)$$

Definition 2.1. *Hamiltonian system (2.6) has an analytic $(n-1)$-dimensional invariant torus with the frequency vector $\boldsymbol{\omega}$ if there exists an analytic canonical transform $\boldsymbol{\vartheta} : (\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\zeta}) \mapsto (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$,*

$$\boldsymbol{x} = \boldsymbol{\xi} + \boldsymbol{u}(\boldsymbol{\xi}), \quad \boldsymbol{y} = \boldsymbol{v}(\boldsymbol{\xi}) + O(|\boldsymbol{\eta}|, |\boldsymbol{\zeta}|), \quad \boldsymbol{z} = \boldsymbol{w}(\boldsymbol{\xi}) + O(|\boldsymbol{\zeta}|),$$
$$\boldsymbol{\xi} \in \mathbb{T}^{n-1}, \quad \boldsymbol{\eta} \in \mathbb{R}^{n-1}, \quad \boldsymbol{\zeta} \in \mathbb{R}^2, \quad (2.7)$$

such that $\boldsymbol{\vartheta}$ puts H into the normal form, i.e.,

$$N(\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\zeta}) \equiv H \circ \boldsymbol{\vartheta} = \boldsymbol{\omega} \cdot \boldsymbol{\eta} + \frac{1}{2}\boldsymbol{\Omega}\boldsymbol{\zeta} \cdot \boldsymbol{\zeta} + o(|\boldsymbol{\eta}|, |\boldsymbol{\zeta}|^2). \quad (2.8)$$

Here $\boldsymbol{\Omega}$ is a constant symmetric matrix. Without loss of generality we may assume that

$$\boldsymbol{\Omega} = \text{diag}(-k, 1), \quad k \in \mathbb{R}, \quad (2.9)$$

and

$$\overline{\boldsymbol{u}} := (2\pi)^{1-n} \int_{\mathbb{T}^{n-1}} \boldsymbol{u}(\boldsymbol{\xi})\, d\boldsymbol{\xi} = 0, \quad \overline{\boldsymbol{w}} = (\alpha, 0), \quad \alpha \in \mathbb{R}.$$

The invariant torus is weakly-hyperbolic if $k \geq 0$. In the normal coordinates system (2.6) reads

$$\dot{\boldsymbol{\xi}} = \boldsymbol{\omega} + O(|\boldsymbol{\eta}|, |\boldsymbol{\zeta}|), \quad \dot{\boldsymbol{\eta}} = O(|\boldsymbol{\eta}|, |\boldsymbol{\zeta}|), \quad \dot{\boldsymbol{\zeta}} = \mathbf{J}\boldsymbol{\Omega}\boldsymbol{\zeta} + O(|\boldsymbol{\eta}|, |\boldsymbol{\zeta}|^2).$$

It has the solution $\boldsymbol{\xi} = \boldsymbol{\omega} t + const, \boldsymbol{\eta} = 0, \boldsymbol{\zeta} = 0$, which defines an invariant torus.

The invariant tori which meet all requirements of Definition 2.1 are known as reducible tori. It was proved in [3] that every analytic hyperbolic invariant torus is reducible.

If $\varepsilon = 0$, then the canonical transformation $\boldsymbol{\vartheta}$ is trivial

$$\boldsymbol{x} = \boldsymbol{\xi}, \quad \boldsymbol{y} = \boldsymbol{\eta}, \quad \boldsymbol{z} = (\alpha, 0) + \boldsymbol{\zeta},$$

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and the normal form is given by

$$
\begin{aligned}
N(\boldsymbol{\eta}, \zeta_2) &\equiv H_0(\boldsymbol{\eta}, \zeta_2) \\
&= \boldsymbol{\omega} \cdot \boldsymbol{\eta} + \frac{1}{2}\zeta_2^2 + \boldsymbol{t}_0 \cdot \boldsymbol{\eta}\zeta_2 + \frac{1}{2}\mathbf{S}_0\boldsymbol{\eta} \cdot \boldsymbol{\eta} + o(|\boldsymbol{\eta}|^2, |\boldsymbol{\eta}||\zeta_2|, |\zeta_2|^2). \quad (2.10)
\end{aligned}
$$

Here $\boldsymbol{t}_0 \in \mathbb{R}^{n-1}$ and the Hessian $\mathbf{S}_0$ are defined by

$$
\boldsymbol{t}_0 = \nabla_y \partial_{z_2} H(0,0), \quad \mathbf{S}_0 = \partial_y^2 H_0(0,0,0). \tag{2.11}
$$

If $\varepsilon = 0$, then $\boldsymbol{\Omega} = \text{diag}\,\{0,1\}$ is degenerate. In other words, the problem of persistence of lower dimensional tori is degenerate in the original Kolmogorov- Melnikov formulation. Introduce the constant matrix

$$
\mathbf{K}_0 = \mathbf{S}_0 - \boldsymbol{t}_0 \otimes \boldsymbol{t}_0,
$$

where the vector $\boldsymbol{t}_0$ and the matrix $\mathbf{S}_0$ are defined by equalities (2.11).

Theorem 2.2. *Let Conditions* (**H.1**) *-* (**H.1**) *be satisfied. Furthermore, assume that*

$$
\mathbf{K}_0\boldsymbol{\eta} \cdot \boldsymbol{\eta} < 0 \quad \text{for all} \ \ \boldsymbol{\eta} \in \mathbb{R}^{n-1} \setminus \{0\}. \tag{2.12}
$$

Then there is $\varepsilon_0 > 0$ such that for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, Hamiltonian system (2.6) has an invariant $(n-1)$-dimensional weakly hyperbolic torus which meets all requirements of Definition 2.1.

Now we can characterize the contents of the paper. In this paper we apply to low-dimensional tori problem a modified version of the variational principle proposed in [16] and developed in [17]. In Section 3 we introduce the special group of canonical transforms which put the Hamiltonina H in the normal form. We investigate in many details the structure of this group and its tangent space.

In the Section 4 we employ the version of the Lyapunov-Schmidt method developed in [19] in order to reduce the problem to the finite system of functional equations named bifurcation equations. To this end we add the modification term $mz_1 + Mz_1^2/2$ to the original Hamiltonian. We obtain the modified Hamiltonian H_m,

$$
H_m(\mathbf{x}, \mathbf{y}, \mathbf{z}) = H_0(\mathbf{y}, z_2) + \varepsilon H_1(\mathbf{x}, \mathbf{y}, \mathbf{z}) + mz_1 + \frac{1}{2}Mz_1^2
$$

Thus we come to the following modified problem. For given $\alpha \in \mathbb{R}^1$ and $k \in [0, 1]$ to find parameters m, M, and a canonical mapping

$$\boldsymbol{\theta} : (\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\zeta}) \to (\mathbf{x}, \mathbf{y}, \mathbf{z}),$$

$$\boldsymbol{x} = \boldsymbol{\xi} + \mathbf{u}(\boldsymbol{\xi}), \quad \mathbf{y} = \mathbf{v} + \mathbf{V}(\boldsymbol{\xi})\boldsymbol{\eta} + \boldsymbol{\Lambda}(\boldsymbol{\xi})\boldsymbol{\zeta} + \frac{1}{2}\boldsymbol{\zeta}^{\top}\mathbf{R}(\boldsymbol{\xi})\boldsymbol{\zeta},$$

$$\mathbf{z} = \mathbf{w}(\boldsymbol{\xi}) + \mathbf{W}(\boldsymbol{\xi})\boldsymbol{\zeta}.$$

which puts the modified hamiltonian in the normal form ,i.e.,

$$H_m \circ \boldsymbol{\theta} = \boldsymbol{\omega}^{\top} \cdot \boldsymbol{\eta} + \frac{1}{2}\boldsymbol{\zeta}^{\top}\boldsymbol{\Omega}\boldsymbol{\zeta} + o(|\boldsymbol{\eta}|, |\boldsymbol{\eta}||\boldsymbol{\zeta}|, |\boldsymbol{\zeta}|^2), \tag{2.13}$$

In this framework α and k are given, while m, M, and e are unknown and should be defined along with a solution. The solvability of the modified problem can be established by using the Nash-Moser implicit function theorem. The obtained solution to the modified problem is a function of α and k. In particular, we have $m = m(\alpha, k)$ and $M = M(\alpha, k)$. Obviously, if $m = M = 0$, then the canonical mapping $\boldsymbol{\theta}$ puts the original hamiltonian $H(\mathbf{x}, \mathbf{y}, \mathbf{z})$ in the normal form. Thus we reduce the original problem to the system of two scalar equations

$$m(\alpha, k) = M(\alpha, k) = 0$$

named the bifurcation equations.

In Sections 5-6 we prove that the modified problem has an analytic solution for all sufficiently small ε. In Sections 7-9 we investigate in details the dependence of solutions to the modified problem on the parameters α and k. We define the Jacobi vector fields associated with the derivatives of these solutions with respect to α and k. We also investigate the properties of the quadratic form of the differential of the action functional. Finally, in Section 10 we prove the existence of the critical point of the action functional and complete the proof of Theorem 2.2.

3   Basic group of of canonical transformations

Recall that our task is to find the canonical transformation (2.7) which puts H into the normal form (2.8). An essential tool in our approach is a special group of canonical transformations. In this section we define such a group as a manifold in the space of analytic mappings and investigate a structure of this manifold.

Canonical transformations Let us consider the totality of all analytic mappings $\boldsymbol{\vartheta} : \mathbb{T}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R}^2 \to \mathbb{T}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R}^2$ of the form

$$\boldsymbol{\vartheta} : (\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\zeta}) \to (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}),$$

$$\boldsymbol{x} = \boldsymbol{\xi} + \mathbf{u}(\boldsymbol{\xi}), \quad \mathbf{y} = \mathbf{v} + \mathbf{V}(\boldsymbol{\xi})\boldsymbol{\eta} + \boldsymbol{\Lambda}(\boldsymbol{\xi})\boldsymbol{\zeta} + \frac{1}{2}\boldsymbol{\zeta}^\top \mathbf{R}(\boldsymbol{\xi})\boldsymbol{\zeta},$$

$$\mathbf{z} = \mathbf{w}(\boldsymbol{\xi}) + \mathbf{W}(\boldsymbol{\xi})\boldsymbol{\zeta}. \tag{3.1}$$

Here $(n-1) \times (n-1)$- matrix valued function $\mathbf{V}$, $(n-1) \times 2$- matrix valued function $\boldsymbol{\Lambda}$, 2×2- matrix values function $\mathbf{W}$, and vector valued functions $\mathbf{u}, \mathbf{v} : \mathbb{T}^{n-1} \to \mathbb{R}^{n-1}$, $\mathbf{v} : \mathbb{T}^{n-1} \to \mathbb{R}^{n-1}$, $\mathbf{w} : \mathbb{T}^{n-1} \to \mathbb{R}^2$, are analytic and 2π periodic in $\boldsymbol{\xi}$, $\mathbf{R}$ is a vector valued quadratic form given by

$$\boldsymbol{\zeta}^\top \mathbf{R} \boldsymbol{\zeta} = \left(\boldsymbol{\zeta}^\top \mathbf{R}_1 \boldsymbol{\zeta}, \dots, \boldsymbol{\zeta}^\top \mathbf{R}_{n-1} \boldsymbol{\zeta}\right)^\top, \quad \mathbf{R}_i = \mathbf{R}_i^\top. \tag{3.2}$$

where 2×2 matrix valued functions $\mathbf{R}_i(\boldsymbol{\xi})$ are analytic and 2π periodic.

Recall that $\boldsymbol{\vartheta} : \mathbb{T}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R}^2 \to \mathbb{T}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R}^2$ is a canonical if $\boldsymbol{\vartheta}^{-1}$ takes solutions of hamiltonian system with a hamiltonian H to solutions of the hamiltonian system with the hamiltonian $H \circ \boldsymbol{\vartheta}$. The mapping $\boldsymbol{\vartheta}$ is a canonical if and only if its Jacobi matrix is symplectic, i.e.,

$$(\boldsymbol{\vartheta}')^\top \mathbf{J}_{2n} \, \boldsymbol{\vartheta}' = \mathbf{J}_{2n}, \tag{3.3}$$

where

$$\mathbf{J}_{2n} = \begin{pmatrix} 0 & \mathbf{I} & 0 \\ -\mathbf{I}_{n-1} & 0 & 0 \\ 0 & 0 & \mathbf{J} \end{pmatrix}, \quad \mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$\mathbf{I}$ is the identity matrix. The following First Structure Theorem gives the complete description of the set of canonical mappings $\boldsymbol{\vartheta} \in \mathcal{A}$.

Theorem 3.1. First Structure Theorem. *Let $\boldsymbol{\vartheta} \in \mathcal{A}$ is given by (3.1). Furthermore assume that the there are analytic inverses $(Id + \mathbf{u})^{-1}$, $\mathbf{V}^{-1}$, and $\mathbf{W}^{-1}$. Then $\boldsymbol{\vartheta}$ is canonical if and only if*

$$\mathbf{V} = (\mathbf{I}_{n-1} + \mathbf{u}')^{-\top}, \tag{3.4a}$$

$$\det \mathbf{W} = 1 \Leftrightarrow \mathbf{W}^\top \mathbf{J} \mathbf{W} = \mathbf{J}, \tag{3.4b}$$

$$\boldsymbol{\Lambda} = -\mathbf{V} (\mathbf{w}')^\top \mathbf{J} \mathbf{W} \equiv -\mathbf{V} \nabla \mathbf{w} \mathbf{J} \mathbf{W}, \tag{3.4c}$$

$$\mathbf{R}_i = -V_{ik} \frac{\partial}{\partial \xi_k} \left(\mathbf{W}^\top\right) \mathbf{J} \mathbf{W}, \tag{3.4d}$$

$$d(\xi_k + u_k) \wedge dv_k + dw_1 \wedge dw_2 = 0. \tag{3.4e}$$

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Moreover, there exist $\beta \in \mathbb{R}^{n-1}$ and 2π -periodic scalar function $\varphi_0(\boldsymbol{\xi})$ such that

$$\mathbf{v} = \beta + \mathbf{V}\left(\nabla\varphi_0 - w_2\nabla w_1\right). \tag{3.4f}$$

Proof. The proof is given in Appendix A. $\qquad\qquad\square$

It is clear that the totality of the canonical mappings $\boldsymbol{\vartheta}$ contains the identity mapping and it is closed with respect to the composition. In other words, it can be regarded as a subgroup of the group of analytical diffeomorphisms. Every mapping $\boldsymbol{\vartheta}$ is completely characterized by the vector $\boldsymbol{\Theta}$ of its coefficients,

$$\boldsymbol{\Theta} = \left(\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{V}, \boldsymbol{\Lambda}, \mathbf{W}, \mathbf{R}_i\right). \tag{3.5}$$

The group structure on the set of the canonical mappings $\boldsymbol{\vartheta}$ induces the group structure on the set of the corresponding vectors $\boldsymbol{\Theta}$. Hence the totality of coefficient vectors $\boldsymbol{\Theta}$ corresponding to the canonical transforms $\boldsymbol{\vartheta}$ can be regarded as a nonlinear manifold in the linear space of all vector-valued analytic 2π-periodic vector-valued functions $\boldsymbol{\Theta}$. We denote this manifold by $\mathcal{G}$. Now our task is to supply $\mathcal{G}$ with a local chart.

In view of Theorem 3.1 the vector $\boldsymbol{\Theta}$ are completely defined by a constant vector β, a scalar function φ_0, vector functions $\mathbf{u}$, $\mathbf{w}$, and by three elements of a symplectic matrix $\mathbf{W}$. We can consider these quantities as local coordinates for the manifold $\mathcal{G}$. More precisely, introduce the vector function

$$\boldsymbol{\varphi} = (\beta, \varphi_0, \mathbf{u}, \mathbf{w}, W_{11}, W_{12}, W_{21}) \tag{3.6}$$

such that the functions $W_{11}, W_{12}, W_{21}, \varphi_0 : \mathbb{T}^{n-1} \to \mathbb{R}$, the vector functions $\mathbf{u} : \mathbb{T}^{n-1} \to \mathbb{R}^{n-1}$, $\mathbf{w} : \mathbb{T}^{n-1} \to \mathbb{R}^2$ are analytic and

$$\overline{\mathbf{u}} \equiv (2\pi)^{1-n}\int_{\mathbb{T}^{n-1}} \mathbf{u}\, d\xi = 0, \quad \overline{\psi_0} \equiv (2\pi)^{1-n}\int_{\mathbb{T}^{n-1}} \psi_0\, d\xi = 0,$$

Next, introduce the mapping $\boldsymbol{\Theta}(\boldsymbol{\varphi})$ defined by the formulae

$$\boldsymbol{\Theta} : \boldsymbol{\varphi} \mapsto \boldsymbol{\Theta}(\boldsymbol{\varphi}) = \left(\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{V}, \boldsymbol{\Lambda}, \mathbf{W}, \mathbf{R}_i\right), \tag{3.7}$$

where

$$\mathbf{V} = (\mathbf{I}_{n-1} + \mathbf{u}')^{-\top}, \quad \mathbf{v} = \beta + \mathbf{V}(\nabla\varphi_0 - w_2\nabla w_1),$$

$$\mathbf{W} = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & \frac{1}{W_{11}}(1 + W_{12}W_{21}) \end{pmatrix}, \tag{3.8}$$

$$\boldsymbol{\Lambda} = -\mathbf{V}(\mathbf{w}')^{\top}\mathbf{J}\mathbf{W}, \quad \mathbf{R}_i = -V_{ik}\left(\frac{\partial}{\partial\xi_k}\mathbf{W}^{\top}\right)\mathbf{J}\,\mathbf{W}.$$

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In view of Theorem 3.1 the mapping $\varphi \mapsto \Theta(\varphi)$ takes the vectors φ to the elements of the manifold $\mathcal{G}$. Hence, this mapping defines a local chart on $\mathcal{G}$. This chart is local since the mapping (3.7) develops singularities at the points where $W_{11} = 0$ and $\det(\mathbf{I}_{n-1} + \mathbf{u}') = 0$.

Tangent space. Denote by $\mathcal{L}$ the linear space of all coordinate vectors φ. Introduce that Gateaux differential

$$D\Theta(\varphi)[\delta\varphi] = \lim_{t \to 0} t^{-1}\big(\Theta(\varphi + t\delta\varphi) - \Theta(\varphi)\big). \tag{3.9}$$

The totality of all vectors

$$\delta\Theta = D\Theta(\varphi)[\delta\varphi], \quad \delta\varphi \in \mathcal{L}, \tag{3.10}$$

can be regarded as the tangential space $\mathrm{Tan}_\Theta \mathcal{G}$ to the manifold $\mathcal{G}$ at the point $\Theta(\varphi)$. Direct calculations show that for every

$$\delta\varphi = (\delta\beta, \delta\varphi_0, \delta\mathbf{u}, \delta\mathbf{w}, \delta W_{11}, \delta W_{12}, \delta W_{21}). \tag{3.11}$$

the vector

$$\delta\Theta = \big(\delta\mathbf{u}, \delta\mathbf{v}, \delta\mathbf{w}, \delta\mathbf{V}, \delta\mathbf{W}, \delta\mathbf{\Lambda}, \delta\mathbf{R}_i\big). \tag{3.12}$$

is defined by the following formulae

$$\delta\mathbf{V} = -\mathbf{V}\,\delta\nabla_\xi\delta\mathbf{u}\,\mathbf{V}, \quad \delta W_{22} = \frac{1}{W_{11}}\big(W_{12}\delta W_{21} + W_{21}\delta W_{12} - W_{22}\delta W_{11}\big),$$

$$\delta\mathbf{W} = \begin{pmatrix} \delta W_{11} & \delta W_{12} \\ \delta W_{21} & \delta W_{22} \end{pmatrix}, \tag{3.13}$$

$$\delta\mathbf{v} = \delta\beta + \mathbf{V}(\nabla\delta\varphi_0 - w_2\nabla\delta w_1 - \delta w_2\nabla w_1) + \delta\mathbf{V}\,(\nabla\varphi_0 - w_2\nabla w_1),$$

$$\delta\mathbf{\Lambda} = \delta\mathbf{V}\,\nabla w\,\mathbf{J}\mathbf{W} - \mathbf{V}\,\nabla(\delta\mathbf{w})\mathbf{J}\mathbf{W} - \mathbf{W}\,\nabla\mathbf{w}\,\mathbf{J}\,\delta\mathbf{W},$$

$$\delta\mathbf{R}_i = -\delta V_{ik}\frac{\partial}{\partial\xi_k}(\mathbf{W}^\top)\mathbf{J}\mathbf{W} - V_{ik}\frac{\partial}{\partial\xi_k}(\delta\mathbf{W}^\top)\mathbf{J}\mathbf{W} - V_{ik}\frac{\partial}{\partial\xi_k}(\mathbf{W}^\top)\mathbf{J}\,\delta\mathbf{W}.$$

The right hand sides of (3.13) are linear differential operators acting on $\delta\varphi$. They can be regarded as the Gâteaux derivatives of the components of $\Theta(\varphi)$ at the point φ in the direction $\delta\varphi$. These relations can be simplified in the particular case when

$$\varphi = \varphi_0(\alpha) \equiv (0, 0, 0, \alpha\mathbf{e}_1, 1, 0, 0).$$

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Notice that for $\varphi = \varphi_0$ we have

$$\mathbf{V} = \mathbf{I}_{n-1}, \quad \mathbf{W} = \mathbf{I}, \quad \mathbf{\Lambda} = 0, \quad \mathbf{R} = 0.$$

We consider this very special case in many details. For $\varphi = \varphi_0$ we will use the special notation for the components of the vector $\delta\varphi$:

$$\mathbf{\Upsilon} \equiv \delta\varphi = (\boldsymbol{\nu}, \psi_0, \boldsymbol{\chi}, \boldsymbol{\lambda}, \Gamma_{11}, \Gamma_{12}, \Gamma_{21}). \tag{3.14}$$

This means that

$$\begin{aligned}
\boldsymbol{\nu} = \delta\boldsymbol{\beta}, \quad \psi_0 = \delta\varphi_0, \quad \boldsymbol{\chi} = \delta\boldsymbol{u}, \quad \boldsymbol{\lambda} = \delta\mathbf{w}, \\
\Gamma_{11} = \delta W_{11}, \quad \Gamma_{12} = \delta W_{12}, \quad \Gamma_{21} = \delta W_{21},
\end{aligned} \tag{3.15}$$

It is easily seen that in this case

$$\mathfrak{Z} := \delta\mathbf{\Theta} \equiv D_\varphi \mathbf{\Theta}(\varphi_0)[\mathbf{\Upsilon}] \tag{3.16}$$

is defined by the equalities

$$\mathfrak{Z} \equiv \Big(\boldsymbol{\chi}, \boldsymbol{\mu}, \boldsymbol{\lambda}, -\nabla\boldsymbol{\chi}, \mathbf{\Gamma}, \nabla(\mathbf{J}\boldsymbol{\lambda}), \partial_{\xi_i}(\mathbf{J}\,\mathbf{\Gamma})\Big), \tag{3.17}$$

where

$$\boldsymbol{\mu} = \boldsymbol{\nu} + \nabla\psi_0, \quad \text{Tr}\,\mathbf{\Gamma} = 0. \tag{3.18}$$

This means that in this case

$$\delta\mathbf{V} = -\nabla_\xi\boldsymbol{\chi}, \quad \delta\mathbf{\Lambda} = -\nabla\boldsymbol{\lambda}\,\mathbf{J} \equiv \nabla(\mathbf{J}\boldsymbol{\lambda}), \quad \mathbf{R}_i = -\frac{\partial}{\partial\xi_i}\mathbf{\Gamma}^\top\mathbf{J} \equiv \frac{\partial}{\partial\xi_i}(\mathbf{J}\mathbf{\Gamma}).$$

$$\delta\mathbf{W} = \mathbf{\Gamma} = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{pmatrix}, \quad \text{where}\;\; \Gamma_{22} = -\Gamma_{11}.$$

In other words, we have the equality $\text{Tan}_{\mathbf{\Theta}(\varphi_0)}\mathcal{G} = \{\mathfrak{Z}\}$. In particular, all tangent spaces to the manifold $\mathcal{G}$ at points $\mathbf{\Theta}(\varphi_0)$ coincide. It is a remarkable fact of the theory that for every φ there is a canonical algebraic isomorphism between the tangent space to the manifold $\mathcal{G}$ at point $\mathbf{\Theta}(\varphi)$ and the tangent space to $\mathcal{G}$ at $\mathbf{\Theta}(\varphi_0)$. This result is given by the following theorem

Theorem 3.2. Second Structure Theorem. (i) *Let $\varphi \in \mathcal{L}$. Let $\mathbf{\Upsilon}$ and $\mathfrak{Z}$ be given by (3.14) and (3.17). Then there exists a vector field*

$$\delta\varphi = (\delta\beta, \delta\varphi_0, \delta\mathbf{u}, \delta\mathbf{w}, \delta W_{11}, \delta W_{12}, \delta W_{21}) \in X_{\sigma, d-1}$$

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such that the corresponding vector of coefficients

$$\delta\boldsymbol{\Theta}(\boldsymbol{\varphi}) = \big(\, \delta\mathbf{u}, \delta\mathbf{v}, \delta\mathbf{w}, \delta\mathbf{V}, \delta\mathbf{W}, \delta\boldsymbol{\Lambda}, \delta\mathbf{R}_i \,\big)$$

given by (3.12)-(3.13) are connected with the vector fields $\boldsymbol{\Upsilon}$ and $\mathfrak{Z}$ by the relations

$$\delta\mathbf{u} \;=\; \chi_i\frac{\partial}{\partial\xi_i}\,(\boldsymbol{\xi} + \mathbf{u}), \tag{3.19a}$$

$$\delta\mathbf{v} \;=\; \mathbf{V}\boldsymbol{\mu} + \boldsymbol{\Lambda}\boldsymbol{\lambda} + \chi_i\,\frac{\partial}{\partial\xi_i}\mathbf{v}, \tag{3.19b}$$

$$\delta\mathbf{w} \;=\; \mathbf{W}\boldsymbol{\lambda} + \chi_i\,\frac{\partial}{\partial\xi_i}\mathbf{w}, \tag{3.19c}$$

$$\delta\mathbf{W} \;=\; \mathbf{W}\boldsymbol{\Gamma} + \chi_i\,\frac{\partial}{\partial\xi_i}\mathbf{W}, \tag{3.19d}$$

$$\delta\mathbf{V} \;=\; -\mathbf{V}\nabla_\xi\boldsymbol{\chi} + \chi_i\,\frac{\partial}{\partial\xi_i}\mathbf{V}, \tag{3.19e}$$

$$\delta\boldsymbol{\Lambda} \;=\; \mathbf{V}\nabla_\xi(\mathbf{J}\boldsymbol{\lambda}) + \chi_i\,\frac{\partial}{\partial\xi_i}\boldsymbol{\Lambda} + \boldsymbol{\lambda}^\top\mathbf{R} + \boldsymbol{\Lambda}\boldsymbol{\Gamma}, \tag{3.19f}$$

$$\delta\mathbf{R}_i \;=\; \frac{\partial}{\partial\xi_i}(\mathbf{J}\boldsymbol{\Gamma}) + \mathbf{R}_i\boldsymbol{\Gamma} + (\mathbf{R}_i\boldsymbol{\Gamma})^\top + \chi_i\,\frac{\partial}{\partial\xi_i}\mathbf{R}_i, \tag{3.19g}$$

$$\delta\boldsymbol{\beta} = \boldsymbol{\nu}, \quad \delta\varphi_0 = \psi_0 + w_2\delta w_1 + \chi_i\,\frac{\partial}{\partial\xi_i}\varphi_0 - w_2\chi_i\,\frac{\partial}{\partial\xi_i}w_1 - \boldsymbol{\nu}\cdot\mathbf{u}. \tag{3.19h}$$

Here $\mathbf{u}$, φ_0, $\mathbf{W}$, $\mathbf{V}$, $\boldsymbol{\Lambda}$, and $\mathbf{R}_i$ are the components of the vector fields $\boldsymbol{\varphi}$ and $\boldsymbol{\Theta}(\boldsymbol{\varphi})$.
(ii) Conversely, let $\delta\boldsymbol{\varphi}$ and $\delta\boldsymbol{\Theta}(\boldsymbol{\varphi})$ be given by the relations (3.11)-(3.13) . Then there exist vector fields

$$\boldsymbol{\Upsilon} = (\boldsymbol{\nu}, \psi_0, \boldsymbol{\chi}, \boldsymbol{\lambda}, \Gamma_{11}, \Gamma_{12}, \Gamma_{21}) \in X_{\sigma, d-1}$$

and

$$\mathfrak{Z} \equiv \Big(\boldsymbol{\chi},\, \boldsymbol{\mu},\, \boldsymbol{\lambda},\, -\nabla\boldsymbol{\chi},\, \boldsymbol{\Gamma},\, \nabla(\mathbf{J}\boldsymbol{\lambda}),\, \partial_{\xi_i}(\mathbf{J}\,\boldsymbol{\Gamma})\Big),$$

which are connected with the components of the vector fields $\delta\boldsymbol{\varphi}$ and $\delta\boldsymbol{\Theta}$ by

the recurrent relations

$$\boldsymbol{\chi} = \mathbf{V}^\top \delta \mathbf{u} \tag{3.20a}$$

$$\boldsymbol{\lambda} = \mathbf{W}^{-1} \delta \mathbf{w} - \chi_i \, \mathbf{W}^{-1} \frac{\partial}{\partial \xi_i} \mathbf{w} \tag{3.20b}$$

$$\boldsymbol{\Gamma} = \mathbf{W}^{-1} \delta \mathbf{W} - \chi_i \, \mathbf{W}^{-1} \frac{\partial}{\partial \xi_i} \mathbf{W}, \tag{3.20c}$$

$$\boldsymbol{\mu} = \mathbf{V}^{-1} \left(\delta \mathbf{W} + \chi_i \frac{\partial}{\partial \xi_i} \mathbf{v} - \boldsymbol{\Lambda} \boldsymbol{\lambda} \right) \tag{3.20d}$$

$$\nabla \psi_0 = \boldsymbol{\mu} - \boldsymbol{\nu}, \quad \boldsymbol{\nu} = \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{T}^{n-1}} \boldsymbol{\mu} \, d\boldsymbol{\xi} = \delta \boldsymbol{\beta}. \tag{3.20e}$$

Proof. The proof is in Appendix B. $\qquad\qquad\square$

4   A modified problem and bifurcation equations

Recall that the main problem is to prove that the Hamiltonian $H(\mathbf{x}, \mathbf{y}, \mathbf{z}) = H_0(\mathbf{y}, z_2) + \varepsilon H_1(\mathbf{x}, \mathbf{y}, \mathbf{z})$ has $(n-1)$-dimensional weakly-hyperbolic invariant torus for all sufficiently small ε. In view of Definition 2.1 it is necessary to find a canonical mapping $\boldsymbol{\theta} : (\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\zeta}) \to (\mathbf{x}, \mathbf{y}, \mathbf{z})$ which puts H in the normal form

$$H \circ \boldsymbol{\theta} = e + \boldsymbol{\omega}^\top \cdot \boldsymbol{\eta} + \frac{1}{2} \boldsymbol{\zeta}^\top \boldsymbol{\Omega} \boldsymbol{\zeta} + o(|\boldsymbol{\eta}|, |\boldsymbol{\eta}||\boldsymbol{\zeta}|, |\boldsymbol{\zeta}|^2), \tag{4.1}$$

where $\boldsymbol{\Omega} = \operatorname{diag}(-k, 1)$, $k \in [0, 1]$. and $e = \operatorname{const}$. For $\varepsilon = 0$ the problem has a family of solutions given by

$$\mathbf{x} = \boldsymbol{\xi}, \quad \mathbf{y} = \boldsymbol{\eta}, \quad \mathbf{z} = \boldsymbol{\alpha} + \boldsymbol{\zeta}, \quad \boldsymbol{\alpha} = (\alpha, 0)^\top, \quad e = k = 0, \quad \alpha \in \mathbb{R}^1.$$

It is easily seen that $(n-1)$- dimensional manifold $\mathbb{T}_\alpha = \{\mathbf{x} = \boldsymbol{\xi}, \mathbf{y} = 0, \mathbf{z} = \boldsymbol{\alpha}\}$ is an invariant torus of the unperturbed system. The totality of these tori forms a foliation of resonance n-dimensional invariant torus, and α can be considered as a label of a leave of this foliation. Notice that α is nothing else but the mean value of z_1 over the invariant torus $\mathbb{T}_\alpha$. Perturbations destroy n-dimensional resonance torus, and only a few $(n-1)$-dimensional invariant tori survive for $\varepsilon \neq 0$. The label α of surviving torus is unknown and should be defined along with a solution to the problem. Therefore, the

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range of unknowns (α, k) is the whole strip $\mathbb{R} \times [0, 1]$. This means that the problem of finding of α and k is not local and can not be solved by using an iteration process. In order to cope with this difficulty we apply the version of the Lyapunov-Schmidt method proposed in [19]. Following [19] we add the modification term $mz_1 + Mz_1^2/2$ to the original Hamiltonian. Thus we get the modified Hamiltonian H

$$H_m(\mathbf{x}, \mathbf{y}, \mathbf{z}) = H_0(\mathbf{y}, z_2) + \varepsilon H_1(\mathbf{x}, \mathbf{y}, \mathbf{z}) + \mathbf{m}^\top \cdot \mathbf{z} + \frac{1}{2}\mathbf{z}^\top \mathbf{M}\mathbf{z} \qquad (4.2)$$

where
$$\mathbf{m} = (m, 0)^\top, \quad \mathbf{M} = \mathrm{diag}\ (M, 0) \qquad (4.3)$$

Consider the following

Modified problem. For given $\alpha \in \mathbb{R}^1$ and $k \in [0, 1]$ to find parameters m, M, e, and a canonical mapping

$$\boldsymbol{\theta} : (\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\zeta}) \to (\mathbf{x}, \mathbf{y}, \mathbf{z}), \qquad (4.4)$$

$$\boldsymbol{x} = \boldsymbol{\xi} + \mathbf{u}(\boldsymbol{\xi}), \quad \mathbf{y} = \mathbf{v} + \mathbf{V}(\boldsymbol{\xi})\boldsymbol{\eta} + \boldsymbol{\Lambda}(\boldsymbol{\xi})\boldsymbol{\zeta} + \frac{1}{2}\boldsymbol{\zeta}^\top \mathbf{R}(\boldsymbol{\xi})\boldsymbol{\zeta},$$

$$\mathbf{z} = \mathbf{w}(\boldsymbol{\xi}) + \mathbf{W}(\boldsymbol{\xi})\boldsymbol{\zeta}. \qquad (4.5)$$

with the following properties. The mapping $\boldsymbol{\theta}$ satisfies the condition

$$\frac{1}{(2\pi)^{n-1}} \int_{\mathbb{T}^{n-1}} w_1(\xi)\, d\xi = \alpha,$$

and puts the modified hamiltonian in the normal form ,i.e.,

$$H_m \circ \boldsymbol{\theta} = e + \boldsymbol{\omega}^\top \cdot \boldsymbol{\eta} + \frac{1}{2}\boldsymbol{\zeta}^\top \boldsymbol{\Omega}\boldsymbol{\zeta} + o(|\boldsymbol{\eta}|, |\boldsymbol{\eta}||\boldsymbol{\zeta}|, |\boldsymbol{\zeta}|^2), \qquad (4.6)$$

We stress that in this framework α and k are given, while m, M, and e are unknown and should be defined along with a solution. The advantage of this approach is that the solvability of the modified problem can be established by using the Nash-Moser implicit function theorem. The obtained solution $\boldsymbol{\theta}, e, m, M$ to the modified problem is a function of α and k. In particular, we have $m = m(\alpha, k)$ and $M = M(\alpha, k)$. Obviously, if $m = M = 0$, then the canonical mapping $\boldsymbol{\theta}$ puts the original hamiltonian $H(\mathbf{x}, \mathbf{y}, \mathbf{z})$ in the normal form. Thus we reduce the original problem to the system of two scalar equations

$$m(\alpha, k) = M(\alpha, k) = 0 \qquad (4.7)$$

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named the bifurcation equations. Hence our first task is to prove the local existence and uniqueness of solutions to the modified problem for all (α, k) and for all sufficiently small ε.

We reduce the modified problem to the system of nonlinear partial differential equations. Notice that in view of Theorem 3.1 the coefficient vectors $\Theta = (\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{V}, \boldsymbol{\Lambda}, \mathbf{W}, \mathbf{R}_i)$ is a function of the coordinate vector

$$\boldsymbol{\varphi} = (\beta, \varphi_0, \mathbf{u}, \mathbf{w}, W_{11}, W_{12}, W_{21}). \tag{4.8}$$

Moreover, the mapping $\boldsymbol{\varphi} \mapsto \boldsymbol{\Theta}(\boldsymbol{\varphi})$ is given by explicit formulae (3.4). Thus relation (4.6) can be regarded as a system of nonlinear equations for $\boldsymbol{\varphi}$. Substituting representation (4.5) with the coefficients $\boldsymbol{\Theta} = \boldsymbol{\Theta}(\boldsymbol{\varphi})$ into (4.6) we arrive at the following system of equations for the vector $\boldsymbol{\varphi}$ given by (4.8).

$$\Phi_1(\boldsymbol{\varphi}, e, m, M) \equiv H_m(\mathrm{id} + \mathbf{u}, \mathbf{v}, \mathbf{w}) - e = 0, \tag{4.9a}$$

$$\Phi_2(\boldsymbol{\varphi}, m, M) \equiv \left\{ \frac{\partial H_m}{\partial \mathbf{y}}(\mathrm{id} + \mathbf{u}, \mathbf{v}, \mathbf{w})\boldsymbol{\Lambda} + \frac{\partial H_m}{\partial \mathbf{z}}(\mathrm{id} + \mathbf{u}, \mathbf{v}, \mathbf{w})\mathbf{W} \right\}^\top = 0, \tag{4.9b}$$

$$\Phi_3(\boldsymbol{\varphi}, m, M) \equiv \left\{ \frac{\partial H_m}{\partial \mathbf{y}}(\mathrm{id} + \mathbf{u}, \mathbf{v}, \mathbf{w})\mathbf{V} \right\}^\top - \boldsymbol{\omega} = 0 \tag{4.9c}$$

$$\Phi_4(\boldsymbol{\varphi}, m, M) \equiv \frac{\partial H_m}{\partial y_i}(\mathrm{id} + \mathbf{u}, \mathbf{v}, \mathbf{w})\mathbf{R}_i + \boldsymbol{\Lambda}^\top \frac{\partial^2 H_m}{\partial \mathbf{y}^2}(\mathrm{id} + \mathbf{u}, \mathbf{v}, \mathbf{w})\boldsymbol{\Lambda} +$$

$$\mathbf{W}^\top \frac{\partial^2 H_m}{\partial \mathbf{z}\partial \mathbf{y}}(\mathrm{id} + \mathbf{u}, \mathbf{v}, \mathbf{w})\boldsymbol{\Lambda} + \left(\mathbf{W}^\top \frac{\partial^2 H_m}{\partial \mathbf{z}\partial \mathbf{y}}(\mathrm{id} + \mathbf{u}, \mathbf{v}, \mathbf{w})\boldsymbol{\Lambda}\right)^\top + \quad (4.9\mathrm{d})$$

$$\mathbf{W}^\top \frac{\partial^2 H_m}{\partial \mathbf{z}^2}(\mathrm{id} + \mathbf{u}, \mathbf{v}, \mathbf{w})\mathbf{W} - \boldsymbol{\Omega} = 0$$

$$\Phi_5(\boldsymbol{\varphi}) \equiv \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{T}^{n-1}} w_1(\xi)\, d\xi - \alpha = 0. \tag{4.9e}$$

Here matrices $\partial^2 H_m/(\partial \mathbf{y})^2$, $\partial^2 H_m/(\partial \mathbf{z}\partial \mathbf{y})$, and $\partial^2 H_m/(\partial \mathbf{z})^2$ have the entries

$$\left(\frac{\partial^2 H_m}{\partial \mathbf{y}^2}\right)_{ij} = \frac{\partial^2 H_m}{\partial y_i \partial y_j}, \quad \left(\frac{\partial H_m}{\partial \mathbf{z}\partial \mathbf{y}}\right)_{pj} = \frac{\partial^2 H_m}{\partial z_p \partial y_j}, \quad \left(\frac{\partial H_m}{\partial \mathbf{z}^2}\right)_{pq} = \frac{\partial H_m}{\partial z_p \partial z_q},$$

$$1 \le i, j \le n-1, \quad 1 \le p, q \le 2,$$

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the row vectors $\partial H_m/\partial \mathbf{y}$, $\partial H_m/\partial \mathbf{z}$ are given by (1.1), i.e.,

$$\frac{\partial H_m}{\partial \mathbf{y}} = (\nabla_y H_m)^\top, \quad \frac{\partial H_m}{\partial \mathbf{z}} = (\nabla_z H_m)^\top$$

The matrices $\mathbf{\Lambda}$, $\mathbf{V}$, $\mathbf{W}$ and the vector $\mathbf{v}$ are expressed in terms of $\boldsymbol{\varphi}$ by equalities (3.8):

$$\mathbf{V} = (\mathbf{I}_{n-1} + \mathbf{u}')^{-\top}, \quad \mathbf{v} = \beta + \mathbf{V}(\nabla\varphi_0 - w_2 \nabla w_1),$$

$$\mathbf{W} = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & \frac{1}{W_{11}}(1 + W_{12}W_{21}) \end{pmatrix}, \tag{4.9f}$$

$$\mathbf{\Lambda} = -\mathbf{V}(\mathbf{w}')^\top \mathbf{J} \mathbf{W}, \quad \mathbf{R}_i = -V_{ik}\left(\frac{\partial}{\partial\xi_k}\mathbf{W}^\top\right)\mathbf{J}\,\mathbf{W}.$$

Notice that the matrices $\mathbf{R}_i$ are symmetric and the operator Φ_4 is a symmetric matrix valued function. Hence Φ_1 is a scalar, Φ_2 is a column vector of dimension 2, Φ_3 is a column vector of dimension $n-1$, Φ_4 is 2×2 symmetric matrix. Therefore, equations (4.9a)-(4.9d) form a system of $(n+5)$ nonlinear differential equations for $n + 5$ functional components of the vector- valued function $\boldsymbol{\varphi}$.

Notice that a solution to the modified problem is not unique. It is easily seen that if a canonical mapping $(\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\zeta}) \to (\mathbf{x}, \mathbf{y}, \mathbf{z})$ puts the hamiltonian H_m to the normal form, then the mapping

$$(\boldsymbol{\xi}', \boldsymbol{\eta}', \boldsymbol{\zeta}') \to (\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\zeta}) \to (\mathbf{x}, \mathbf{y}, \mathbf{z}),$$

$$\boldsymbol{\xi} = \boldsymbol{\xi}' + c, \quad \boldsymbol{\eta} = \boldsymbol{\eta}', \quad \boldsymbol{\zeta} = \begin{pmatrix} (1 + kb^2)^{1/2} & b \\ kb & (1 + kb^2)^{1/2} \end{pmatrix}\boldsymbol{\zeta}',$$

$c \in \mathbb{R}^{n-1}$, $b \in \mathbb{R}$, also puts H_m in the normal form. Hence the modified problem has n-parametric family of solutions. In order to eliminate this arbitrariness, we impose the orthogonality conditions

$$\int_{\mathbb{T}^{n-1}} \mathbf{u}(\boldsymbol{\xi})\,d\xi = 0, \quad \int_{\mathbb{T}^{n-1}} W_{12}(\boldsymbol{\xi})\,d\xi = 0. \tag{4.9g}$$

Recall that by definition of the local coordinate on the manifold $\mathcal{G}$, the function φ_0 in (4.8) has zero mean. Thus we have to add the orthogonality condition

$$\int_{\mathbb{T}^{n-1}} \varphi_0(\boldsymbol{\xi})\,d\xi = 0. \tag{4.9h}$$

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Equations (4.9a)-(4.9e) *and conditions* (4.9g)-(4.9h) *form a closed system of equations for* $\boldsymbol{\varphi}$ *and parameters* m, M, e.

Throughout of the paper we will use the following consequence of this equations. Introduce the important differential operator

$$\boldsymbol{\partial} := \boldsymbol{\omega}^{\top} \cdot \nabla_{\xi} \equiv \omega_i \frac{\partial}{\partial \xi_i}. \tag{4.10}$$

Lemma 4.1. *If* $\boldsymbol{\varphi}$ *is an analytic solution to equations* (4.9a)- (4.9d) *and* $\mathbf{v}$ *is given by* (4.9f), *then*

$$\boldsymbol{\omega} + \boldsymbol{\partial}\mathbf{u} = \nabla_y H_m(\boldsymbol{\xi} + \mathbf{u}, \mathbf{v}, \mathbf{w}),$$
$$\boldsymbol{\partial}\mathbf{v} = -\nabla_x H_m(\boldsymbol{\xi} + \mathbf{u}, \mathbf{v}, \mathbf{w}), \quad \boldsymbol{\partial}\mathbf{w} = \mathbf{J}\nabla_z H_m(\boldsymbol{\xi} + \mathbf{u}, \mathbf{v}, \mathbf{w}). \tag{4.11}$$

The proof is in Appendix C.

Operator equation. Now we reduce system of differential equations (4.9) to the operator equation in Banach spaces of analytic functions. Notice that unknowns are the vector-valued function $\boldsymbol{\varphi}$ and the constants e, m, and M. Equations (4.9) also depend on the parameters α and k. It is convenient to introduce the vectors

$$\mathfrak{f} = (\alpha, k), \quad \mathfrak{u} = (\boldsymbol{\varphi}, e, m, M) \tag{4.12}$$

where $\boldsymbol{\varphi} = (\boldsymbol{\beta}, \varphi_0, \mathbf{u}, \mathbf{w}, W_{11}, W_{12}, W_{22})$.

Now we introduce some notation which is used throughout the rest of the paper.

Definition 4.2. *Let* $\varrho > 0$ *be given by Condition* (**H.1**). *Denote by* Σ_{ϱ} *the slab*

$$\Sigma_{\varrho} = \big\{ (\alpha, k) : \text{ Re } \alpha \in \mathbb{R}, |\text{Im } \alpha| \leq \varrho, \text{ Re } k \in [0, 1], |\text{Im } k| \leq \varrho \big\} \tag{4.13}$$

Definition 4.3. *Let* $\varrho > 0$ *bi given by Condition* (**H.1**). *For every* $\sigma \in [0, 1]$ *and any integer* $d \geq 0$ *denote by* $\mathcal{A}_{\sigma,d}$ *the Banach space of all functions*

$$\mathbf{u} : \Re_{\sigma\varrho} \to \mathbb{C}, \quad \Re_{\sigma\varrho} = \big\{ \boldsymbol{\xi} : \text{ Re } \boldsymbol{\xi} \in \mathbb{T}^{n-1}, \ |\text{Im } \boldsymbol{\xi}| \leq \sigma\varrho \big\},$$

with the finite norm

$$\|u\|_{\sigma,d} = \max_{0 \leq |k| \leq d} \sup_{\boldsymbol{\xi} \in \Re_{\sigma\varrho}} |\partial^k u(\boldsymbol{\xi})|. \tag{4.14}$$

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Notice that system (4.9) consists of differential equations (4.9a)-(4.9d) and orthogonality conditions (4.9e)-(4.9g). It is convenient to incorporate the orthogonality conditions in the definition of the Banach spaces. Thus we arrive at the following

Definition 4.4. *For every $\sigma \in [0,1]$ and any integer $d \geq 0$ denote by $\mathcal{E}_{\sigma,d}$ the subspace of the Banach space $\mathbb{C}^{n-1} \times \mathcal{A}_{\sigma,d} \times \mathcal{A}_{\sigma,d}^{n-1} \times \mathcal{A}_{\sigma,d}^2 \times \mathcal{A}_{\sigma,d}^3 \times \mathbb{C}^3$ which consists of all vectors*

$$\mathfrak{u} = (\boldsymbol{\varphi}, e, m, M) \quad with \quad \boldsymbol{\varphi} = (\boldsymbol{\beta}, \varphi_0, \mathbf{u}, \mathbf{w}, W_{11}, W_{12}, W_{21}),$$

satisfying the orthogonality conditions (4.9g)-(4.9h).

Definition 4.5. *For every $\sigma \in [0,1]$ and any integer $d \geq 0$ denote by $\mathcal{F}_{\sigma,d}$ the subspace of $\mathcal{A}_{\sigma,d} \times \mathcal{A}_{\sigma,d}^{n-1} \times \mathcal{A}_{\sigma,d}^2 \times \mathcal{A}_{\sigma,d}^4 \times \mathbb{C}$ which consists of all vectors*

$$\boldsymbol{F} = (F_1, F_2, F_3, F_4, f_5)$$

such that F_4 is a symmetric 2×2-matrix valued function..

Introduce the nonlinear operator

$$\boldsymbol{\Phi}(\mathfrak{u}, \mathfrak{f}) = \\ \big(\Phi_1(\boldsymbol{\varphi}, e, m, M), \quad \Phi_2(\boldsymbol{\varphi}, m, M), \Phi_3(\boldsymbol{\varphi}, m, M), \Phi_4(\boldsymbol{\varphi}, m, M; k), \Phi_5(\mathbf{w}; \alpha) \big), \tag{4.15}$$

where Φ_i are given by (4.9). It is worthy noting that $\boldsymbol{\Phi}$ is a linear function of the scalars e, m, M, and k, α. Hence the modified problem (4.9) can be written in the form of the operator equation

$$\boldsymbol{\Phi}(\mathfrak{f}, \mathfrak{u}) = 0. \tag{4.16}$$

Remark For $\varepsilon = 0$ and $\mathfrak{f} \in \mathbb{C}^2$, equation (4.16) has the unique solution

$$\mathfrak{v}(\mathfrak{f}) = (\varphi_0(\alpha), -k\alpha - k\alpha^2, -k\alpha, -k), \tag{4.17}$$

where

$$\varphi_0(\alpha) = (0, 0, \alpha \mathbf{e}_1, 1, 0, 0)$$

Notice that Φ is a nonlinear analytic differential operator. The general theory of the nonlinear differential operators in spaces of analytic functions $\mathcal{A}_{\sigma,d}$ is considered in the monograph [15] and [20]. We use the following proposition, see [15] Ch.6, which constitute the continuity and differentiability of the operator Φ.

Proposition 4.6. *Let $\varrho > 0$ be given by Condition (**H.1**). For every $\sigma \in [0,1]$ and $d \geq 1$, there are $r > 0$ and $c > 0$ with the properties. If $\mathfrak{f} \in \Sigma_\varrho$ and $\mathfrak{u} \in \mathcal{E}_{\sigma,d}$ satisfy*

$$\|\varphi - \varphi_0(\alpha)\|_{\sigma,d} \leq r, \quad \varphi_0 = (0,0,0,\alpha e_1, 1, 0, 0),$$

then $\Phi(\mathfrak{f}, \mathfrak{u}) \in \mathcal{F}_{\sigma,0}$ and

$$\|\Phi(\mathfrak{f}, \mathfrak{u})\|_{\sigma,0} \leq c + c(|e| + |m| + |M|).$$

If in addition

$$\|\varphi + \delta\varphi - \varphi_0(\alpha)\|_{\sigma,d} \leq r$$

then

$$\Phi_i(\varphi + \delta\varphi, e, m, M) - \Phi_i(\varphi, e, m, M) = $$
$$D_\varphi \Phi_i(\varphi, e, m, M)[\delta\varphi] + Q_{\Phi_i}(\varphi, \delta\varphi, e, m, M).$$

The linear operator $D_\varphi \Phi_i$ and the remainder Q_{Φ_i} admit the estimates

$$\|D_\varphi \Phi_i(\varphi, e, m, M)[\delta\varphi]\|_{\sigma,d-1} \leq c(1 + |e| + |m| + |M|) \, \|\delta\varphi\|_{\sigma,d}, \quad (4.18)$$

$$\|Q_{\Phi_i}(\varphi, \delta\varphi, e, m, M)\|_{\sigma,d-1} \leq c(1 + |e| + |m| + |M|) \, \|\delta\varphi\|_{\sigma,d}^2. \quad (4.19)$$

Third structural theorem. In order to apply the Nash-Moser implicit function theorem to operator equation (4.16), we have to prove that the linear operator $D_\mathfrak{u}\Phi$ has an approximate inverse. In other words, we have to show that for every

$$\mathbf{F} = (F_1, F_2, F_3, F_4, f_5) \in \mathcal{F}_{\sigma,d} \quad (4.20)$$

the linear equation

$$D_\varphi \Phi(\varphi)[\delta\varphi] + D_e \Phi(\varphi)[\delta e] + D_m \Phi(\varphi)[\delta m] + D_M \Phi(\varphi)[\delta M] = \mathbf{F} \quad (4.21)$$

has an analytic approximate solution $(\delta\varphi, e, m, M)$. This problem looks like very difficult because of the complexity of the expression for $\delta\Phi = D_\varphi \Phi(\varphi)[\delta\varphi]$. The remarkable fact of KAM theory is that linear equation (4.21) can be reduced to the triangle system of first order differential equations with constant coefficients in the principle part. This very special change of variables is given by the second structural Theorem 3.2. In order to define,

it choose an arbitrary vector $\delta\boldsymbol{\varphi} = (\delta\boldsymbol{\beta}, \delta\varphi_0, \delta\mathbf{u}, \delta\mathbf{w}, \delta W_{11}, \delta W_{12}, \delta W_{21})$. Next notice that relations (3.20) in Theorem 3.2 read

$$\boldsymbol{\chi} = \mathbf{V}^\top \delta\mathbf{u} \tag{4.22a}$$

$$\boldsymbol{\lambda} = \mathbf{W}^{-1}\delta\mathbf{w} - \chi_i\,\mathbf{W}^{-1}\frac{\partial}{\partial\xi_i}\mathbf{w} \tag{4.22b}$$

$$\boldsymbol{\Gamma} = \mathbf{W}^{-1}\delta\mathbf{W} - \chi_i\,\mathbf{W}^{-1}\frac{\partial}{\partial\xi_i}\mathbf{W}, \tag{4.22c}$$

$$\boldsymbol{\mu} = \mathbf{V}^{-1}\Big(\delta\mathbf{v} + \chi_i\,\frac{\partial}{\partial\xi_i}\mathbf{v} - \boldsymbol{\Lambda}\boldsymbol{\lambda}\Big) \tag{4.22d}$$

$$\nabla\psi_0 = \boldsymbol{\mu} - \delta\boldsymbol{\beta}, \quad \delta\boldsymbol{\beta} = \frac{1}{(2\pi)^{n-1}}\int_{\mathbb{T}^{n-1}}\boldsymbol{\mu}\,d\boldsymbol{\xi}, \tag{4.22e}$$

where $\delta\mathbf{W}$ is given in terms of $\delta\boldsymbol{\varphi}$ by (3.11). These equalities completely define the vector

$$\boldsymbol{\Upsilon} = (\delta\boldsymbol{\beta}, \psi_0, \boldsymbol{\chi}, \boldsymbol{\lambda}, \Gamma_{11}, \Gamma_{12}, \Gamma_{21}).$$

Recall that $\boldsymbol{\Gamma}$ is 2×2 matrix such that $\Gamma_{22} = -\Gamma_{11}$. It is worth noting that equalities (4.22) establishes linear algebraic relations between $\delta\boldsymbol{\varphi}$ and $\boldsymbol{\Upsilon}$ The following theorem shows that the change of variables (4.22) brings the linear operator $D_{\boldsymbol{\varphi}}\boldsymbol{\Phi}$ to the simple canonical form.

Theorem 4.7. Third Structure Theorem. *Let $\boldsymbol{\varphi}$, α and k meet all requirements of Proposition 4.6. Furthermore assume that*

$$\delta\Phi_i = D_{\boldsymbol{\varphi}}\Phi_i(\boldsymbol{\varphi})[\delta\boldsymbol{\varphi}]$$

Then for every $\delta\boldsymbol{\varphi} \in \mathbb{C}^{n-1} \times \mathcal{A}_{\sigma,d} \times \mathcal{A}_{\sigma,d}^{n-1} \times \mathcal{A}_{\sigma,d}^2 \times \mathcal{A}_{\sigma,d}^3,$

$$\delta\Phi_1 = \boldsymbol{\partial}\psi_0 + \boldsymbol{\omega}^\top \cdot \delta\boldsymbol{\beta} + \Pi_1[\boldsymbol{\chi}, \boldsymbol{\mu}, \boldsymbol{\lambda}], \tag{4.23a}$$

$$\delta\Phi_2 = \mathbf{J}\boldsymbol{\partial}\boldsymbol{\lambda} + \boldsymbol{\Omega}\boldsymbol{\lambda} + \mathbf{T}\boldsymbol{\mu} + \Pi_2[\boldsymbol{\chi}, \boldsymbol{\lambda}, \boldsymbol{\Gamma}] \tag{4.23b}$$

$$\delta\Phi_3 = -\boldsymbol{\partial}\boldsymbol{\chi} + \mathbf{S}\boldsymbol{\mu} + \mathbf{T}^\top\boldsymbol{\lambda} + \Pi_3[\boldsymbol{\chi}], \tag{4.23c}$$

$$\delta\Phi_4 = \boldsymbol{\partial}(\mathbf{J}\boldsymbol{\Gamma}) + \boldsymbol{\Omega}\boldsymbol{\Gamma} + (\boldsymbol{\Omega}\boldsymbol{\Gamma})^\top + \mathbf{U}_{ij}\frac{\partial\lambda_i}{\partial\xi_j} + \mu_i\mathbf{E}_i + \lambda_i\mathbf{K}_i + \Pi_4[\boldsymbol{\chi}, \boldsymbol{\Gamma}] \tag{4.23d}$$

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$$\delta \Phi_5 = \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{T}^{n-1}} \left((\mathbf{W}\boldsymbol{\lambda})_1 + \chi_i \frac{\partial w_1}{\partial \xi_i} \right) d\boldsymbol{\xi}. \tag{4.23e}$$

Here $\delta\boldsymbol{\beta}, \psi_0, \boldsymbol{\chi}, \boldsymbol{\lambda}, \Gamma_{11}, \Gamma_{12}, \Gamma_{21}$ and $\delta\boldsymbol{\varphi}$ are interrelated by relations (4.22) . The linear differential operators Π_i are defined by

$$\Pi_1[\boldsymbol{\chi}, \boldsymbol{\mu}, \boldsymbol{\mu}, \boldsymbol{\lambda}] = \frac{\partial \Phi_1}{\partial \xi_i} \chi_i + \Phi_2^\top \cdot \boldsymbol{\lambda} + \Phi_3^\top \cdot \boldsymbol{\mu}, \tag{4.24a}$$

$$\Pi_2[\boldsymbol{\chi}, \boldsymbol{\lambda}, \boldsymbol{\Gamma}] = \chi_i \frac{\partial \Phi_2}{\partial \xi_i} + \boldsymbol{\Gamma}^\top \Phi_2 + \Phi_4 \boldsymbol{\lambda} + (\mathbf{J}\boldsymbol{\lambda})'_\xi \Phi_3, \tag{4.24b}$$

$$\Pi_3[\boldsymbol{\chi}] = \chi_i \frac{\partial \Phi_3}{\partial \xi_i} - \boldsymbol{\chi}'_\xi \Phi_3, \tag{4.24c}$$

$$\Pi_4[\boldsymbol{\chi}, \boldsymbol{\Gamma}] = \chi_i \frac{\partial \Phi_4}{\partial \xi_i} + \Phi_{3,i} \frac{\partial}{\partial \xi_i}(\mathbf{J}\boldsymbol{\Gamma}) + \Phi_4 \boldsymbol{\Gamma} + (\Phi_4 \boldsymbol{\Gamma})^\top, \tag{4.24d}$$

where $\Phi_i = \Phi_i(\boldsymbol{\varphi}, e, m, M)$. The matrix $\mathbf{S}$ and $\mathbf{T}$ are given in terms of $\boldsymbol{\varphi}$ by the equalities

$$\mathbf{S} = \mathbf{V}^\top \frac{\partial^2 H}{\partial \mathbf{y}^2}(\boldsymbol{\xi} + \mathbf{u}, \mathbf{v}, \mathbf{w})\mathbf{V},$$

$$\mathbf{T} = \mathbf{W}^\top \frac{\partial^2 H}{\partial \mathbf{z} \partial \mathbf{y}}(\boldsymbol{\xi} + \mathbf{u}, \mathbf{v}, \mathbf{w})\mathbf{V} + \boldsymbol{\Lambda}^\top \frac{\partial^2 H}{\partial \mathbf{y}^2}(\boldsymbol{\xi} + \mathbf{u}, \mathbf{v}, \mathbf{w})\mathbf{V}. \tag{4.25}$$

They admit the estimates

$$\|\mathbf{S} - \mathbf{S}_0\|_{\sigma,d-1} + \|\mathbf{T} - \mathbf{T}_0\|_{\sigma,d-1} \le c(\varepsilon + r), \tag{4.26}$$

where c depends only on σ, d and H, the constant matrices $\mathbf{S}_0$, $\mathbf{T}_0$ are given by

$$\mathbf{S}_0 = \frac{\partial^2 H_0}{\partial \mathbf{y}^2}(0,0), \quad \mathbf{T}_0 = \frac{\partial^2 H_0}{\partial \mathbf{z} \partial \mathbf{y}}(0,0) \tag{4.27}$$

The symmetric matrix valued functions $\mathbf{U}_{ij}$, $\mathbf{E}_i$, $\mathbf{K}_i$ admit the estimates

$$\|\mathbf{U}_{ij}\|_{\sigma,d-1} + \|\mathbf{E}_i\|_{\sigma,d-1} + \|\mathbf{K}_i\|_{\sigma,d-1} \le c. \tag{4.28}$$

Proof. The proof is in Appendix D. □

Theorem 4.7 gives the representation of the derivatives $D_{\boldsymbol{\varphi}} \boldsymbol{\Phi} = \delta \boldsymbol{\Phi}$. As corollary of this result we obtain the representation for the full differential $D_{\mathfrak{u}} \boldsymbol{\Phi}(\mathfrak{f}, \mathfrak{u})$ defined by

$$D_{\mathfrak{u}} \boldsymbol{\Phi}(\mathfrak{f}, \mathfrak{u})[\delta \mathfrak{u}] = \lim_{\tau \to 0} \frac{1}{\tau} \big(\boldsymbol{\Phi}(\mathfrak{f}, \mathfrak{u} + \tau \delta \mathfrak{u}) - \boldsymbol{\Phi}(\mathfrak{f}, \mathfrak{u}) \big). \tag{4.29}$$

The result is given by the following

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Corollary 4.8. *Under the assumptions of Theorem 4.7 the operators $D_{\mathfrak{u}}\Phi_i$ admit the representation*

$$D_{\mathfrak{u}}\Phi_1(\mathfrak{f},\mathfrak{u})[\delta\mathfrak{u}] \equiv \tag{4.30a}$$

$$\boldsymbol{\partial}\psi_0 + \boldsymbol{\omega}^\top \cdot \delta\boldsymbol{\beta} + \delta e + \delta m \cdot w_1 + \frac{1}{2}\delta M w_1^2 + \Pi_1[\boldsymbol{\chi},\boldsymbol{\mu},\boldsymbol{\lambda}],$$

$$D_{\mathfrak{u}}\Phi_2(\mathfrak{f},\mathfrak{u})[\delta\mathfrak{u}] \equiv \tag{4.30b}$$

$$\mathbf{J}\boldsymbol{\partial}\boldsymbol{\lambda} + \boldsymbol{\Omega}\boldsymbol{\lambda} + \mathbf{T}\boldsymbol{\mu} + \delta m \mathbf{W}^\top \mathbf{e}_1 + \delta M w_1 \mathbf{W}^\top \mathbf{e}_1 + \Pi_2[\boldsymbol{\chi},\boldsymbol{\lambda},\boldsymbol{\Gamma}],$$

$$\boldsymbol{\mu} = \nabla\psi_0 + \delta\boldsymbol{\beta}$$

$$D_{\mathfrak{u}}\Phi_3(\mathfrak{f},\mathfrak{u})[\delta\mathfrak{u}] \equiv \tag{4.30c}$$

$$-\boldsymbol{\partial}\boldsymbol{\chi} + \mathbf{S}\boldsymbol{\mu} + \mathbf{T}^\top\boldsymbol{\lambda} + \Pi_3[\boldsymbol{\chi}],$$

$$D_{\mathfrak{u}}\Phi_4(\mathfrak{f},\mathfrak{u})[\delta\mathfrak{u}] \equiv \tag{4.30d}$$

$$\boldsymbol{\partial}(\mathbf{J}\boldsymbol{\Gamma}) + \boldsymbol{\Omega}\boldsymbol{\Gamma} + (\boldsymbol{\Omega}\boldsymbol{\Gamma})^\top + \mathbf{U}_{ij}\frac{\partial\lambda_i}{\partial\xi_j} + \mu_i\mathbf{E}_i + \lambda_i\mathbf{K}_i +$$

$$\mathbf{W}^\top\delta M \mathbf{W} + \Pi_4[\boldsymbol{\chi},\boldsymbol{\Gamma}],$$

$$D_{\mathfrak{u}}\Phi_5(\mathfrak{f},\mathfrak{u})[\delta\mathfrak{u}] \equiv \tag{4.30e}$$

$$\frac{1}{(2\pi)^{n-1}}\int_{\mathbb{T}^{n-1}}\left((\mathbf{W}\boldsymbol{\lambda})_1 + \chi_i\frac{\partial w_1}{\partial\xi_i}\right)d\boldsymbol{\xi}$$

Proof. Relations (4.9), (4.15) lead to the following expressions for $D_e\Phi$, $D_m\Phi$, and $D_M\Phi$

$$D_e\Phi_1[\delta e] = \delta e, \quad D_e\Phi_i[\delta e] = 0 \quad \text{for} \quad i = 2,3,4,5$$

$$D_m\Phi_1[\delta m] = \delta m w_1, \quad D_m\Phi_2[\delta m] = \delta m \mathbf{W}^\top\mathbf{e}_1,$$

$$D_m\Phi_3[\delta m] = 0, \quad D_m\Phi_4[\mathbf{m}] = 0, \quad D_m\Phi_5[\delta m] = 0,$$

$$D_M\Phi_1[\delta M] = \frac{1}{2}w_1^2\delta M, \quad D_M\Phi_2[\delta M] = \mathbf{W}^\top\delta M\mathbf{w},$$

$$D_M\Phi_3[\delta M] = 0, \quad D_M\Phi_4[\delta M] = \mathbf{W}^\top\delta M\mathbf{W}, \quad D_M\Phi_5[\delta M] = 0.$$

Substituting these expressions into the identity

$$D_{\mathfrak{u}}\boldsymbol{\Phi}(\mathfrak{f},\mathfrak{u})[\delta\mathfrak{u}] = \delta\boldsymbol{\Phi} + D_e\boldsymbol{\Phi}(\mathfrak{f},\mathfrak{u})\delta e + D_m\boldsymbol{\Phi}(\mathfrak{f},\mathfrak{u})\delta m + D_M\boldsymbol{\Phi}(\mathfrak{f},\mathfrak{u})\delta M$$

and recalling expressions (4.23) for $\delta\boldsymbol{\Phi}_i$ in Theorem 4.7 we arrive at the desired identity (4.30). $\square$

5   Linear Problem

In this section we prove the existence an approximate inverse of the Gateux derivative of the operator $\boldsymbol{\Phi}$. This result plays the crucial role in the proof of solvability of the modified problem (4.9). Choose an arbitrary $\sigma \in [1/2, 1]$ and an arbitrary integer $d \geq 1$. Next, fix $\mathfrak{f} = (\alpha, k) \in \Sigma_\varrho$. In view of Corollary 4.8 the operator $\boldsymbol{\Phi}(\mathfrak{f}, \mathfrak{u})$ is differentiable with respect to $\mathfrak{u}$ for all $\mathfrak{u} = (\boldsymbol{\varphi}, e, m, M)$ satisfying the condition

$$\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_0(\alpha)\|_{\sigma,d} \leq r.$$

Notice that $\boldsymbol{\Phi}$ is a linear function of the parameters e, m, M. Consider the linear operator equation

$$D_\mathfrak{u}\boldsymbol{\Phi}(\mathfrak{f}, \mathfrak{u})[\delta\mathfrak{u}] = \mathbf{F} \in \mathcal{F}_{\sigma,d}. \tag{5.1}$$

We are looking for a solution

$$\delta\mathfrak{u} = (\delta\boldsymbol{\varphi}, \delta e, \delta m, \delta M)$$

to this equation in the space $\mathcal{E}_{\sigma,d}$ given by Definition 4.4. Hence $\delta\boldsymbol{\varphi}$ have to satisfy the orthogonality conditions

$$\int_{\mathbb{T}^{n-1}} \delta\varphi_0 \, d\xi = 0, \quad \int_{\mathbb{T}^{n-1}} \delta\mathbf{u} \, d\xi = 0, \quad \int_{\mathbb{T}^{n-1}} \delta W_{12} \, d\xi = 0, \tag{5.2}$$

which are similar condition (4.9h)-(4.9g) for $\boldsymbol{\varphi}$. Relations (5.1)-(5.2) form the closed system of equations for the vector $\delta\mathfrak{u}$. However, the operator $D_\mathfrak{u}\boldsymbol{\Phi}$ has the complicated structure and its type is indefinite. In order to simplify the equation, we exploit Corollary 4.8 of Third Structural Theorem 4.7 and reduce equation (5.1) to the triangular canonical form with constant coefficients in the principal part. To this end we make the linear change of unknown functions and introduce the new unknown vector field $\Upsilon = (\delta\boldsymbol{\beta}, \psi_0, \boldsymbol{\lambda}, \boldsymbol{\chi}, \boldsymbol{\Gamma},)$ which is connected with

$$\delta\boldsymbol{\varphi} = (\delta\boldsymbol{\beta}, \delta\varphi_0, \delta\mathbf{u}, \delta\mathbf{w}, \delta\mathbf{W})$$

by relations (4.22). Using identities (4.30) we rewrite equations (5.1)-(5.2) in the equivalent form

$$\boldsymbol{\partial}\psi_0 + \boldsymbol{\omega}^\top \cdot \delta\boldsymbol{\beta} + \Pi_1[\boldsymbol{\chi}, \boldsymbol{\mu}, \boldsymbol{\lambda}] + \delta e + \delta\mathbf{m} \cdot \mathbf{w} + \frac{1}{2}\mathbf{w}^\top \delta\mathbf{M}\mathbf{w} = F_1, \quad (5.3a)$$

$$\mathbf{J}\boldsymbol{\partial}\boldsymbol{\lambda} + \mathbf{T}\boldsymbol{\mu} + \Pi_2[\boldsymbol{\chi}, \boldsymbol{\lambda}, \boldsymbol{\Gamma}] + \mathbf{W}^\top \delta\mathbf{m} + \mathbf{W}^\top \delta\mathbf{M}\mathbf{w} = F_2, \quad (5.3b)$$

$$\boldsymbol{\mu} = \nabla\psi_0 + \delta\boldsymbol{\beta}$$

$$-\boldsymbol{\partial}\boldsymbol{\chi} + \mathbf{S}\boldsymbol{\mu} + \mathbf{T}^\top\boldsymbol{\lambda} + \Pi_3[\boldsymbol{\chi}] = F_3, \quad (5.3c)$$

$$\boldsymbol{\partial}(\mathbf{J}\boldsymbol{\Gamma}) + \boldsymbol{\Omega}\boldsymbol{\Gamma} + (\boldsymbol{\Omega}\boldsymbol{\Gamma})^\top + \mathbf{U}_{ij}\frac{\partial\lambda_i}{\partial\xi_j} + \mu_i\mathbf{E}_i + \lambda_i\mathbf{K}_i \quad (5.3d)$$

$$+\mathbf{W}^\top\delta\mathbf{M}\mathbf{W} + \Pi_4[\boldsymbol{\chi}, \boldsymbol{\Gamma}] = F_4.$$

$$(2\pi)^{1-n}\int_{\mathbb{T}^{n-1}}\left((\mathbf{W}\boldsymbol{\lambda})_1 + \chi_i\frac{\partial w_1}{\partial\xi_i}\right)d\boldsymbol{\xi} = f_5, \quad (5.3e)$$

$$(2\pi)^{1-n}\int_{\mathbb{T}^{n-1}}\left[\psi_0 - \delta\boldsymbol{\beta}\cdot\mathbf{u} + w_2(\mathbf{W}\boldsymbol{\lambda})_1\right]d\xi = 0 \quad (5.3f)$$

$$(2\pi)^{1-n}\int_{\mathbb{T}^{n-1}}\mathbf{V}^{-\top}\boldsymbol{\chi}\,d\xi = 0, \quad \int_{\mathbb{T}^{n-1}}(\mathbf{W}\boldsymbol{\Gamma})_{12}\,d\xi = 0. \quad (5.3g)$$

Recall that the matrices $\mathbf{S}$ and $\mathbf{T}$ are given by (4.25). This matrices along with the matrices $\mathbf{U}$, $\mathbf{E}_i$, and $\mathbf{K}_i$ admit estimates (4.27) and (4.28). The linear operators Π are defined by formulae (4.24). They vanishes when $\Phi = 0$.

5.1 Approximate equations

Relations (5.3) form the closed system of equations and the orthogonality conditions for the functions ψ_0, $\boldsymbol{\chi}$, $\boldsymbol{\lambda}$, $\boldsymbol{\Gamma}$, and the parameters $\boldsymbol{\beta}$, δm, δM, and δe. This system is not in a triangle form and it is inconvenient for the investigation. In order to cope with this difficulty we notice that, in accordance with the basic principles of the KAM theory, we are looking for an approximate solution to equations (5.3). This approximate solutions should satisfy equations (5.3) with the accuracy up to the discrepancy. Hence we can omit the operators Π_i in (5.3). Thus we arrive at the approximate

equations

$$\boldsymbol{\partial}\psi_0 + \boldsymbol{\omega}^\top \cdot \delta\boldsymbol{\beta} + \delta e + \delta\mathbf{m} \cdot \mathbf{w} + \frac{1}{2}\mathbf{w}^\top \delta\mathbf{Mw} = F_1, \qquad (5.4\text{a})$$

$$\mathbf{J}\boldsymbol{\partial}\boldsymbol{\lambda} + \boldsymbol{\Omega}\boldsymbol{\lambda} + \mathbf{T}\boldsymbol{\mu} + \mathbf{W}^\top \delta\mathbf{m} + \mathbf{W}^\top \delta\mathbf{Mw} = F_2, \qquad (5.4\text{b})$$

$$\boldsymbol{\mu} = \nabla\psi_0 + \delta\boldsymbol{\beta}$$

$$-\boldsymbol{\partial}\boldsymbol{\chi} + \mathbf{S}\boldsymbol{\mu} + \mathbf{T}^\top \boldsymbol{\lambda} = F_3, \qquad (5.4\text{c})$$

$$\boldsymbol{\partial}(\mathbf{J}\boldsymbol{\Gamma}) + \boldsymbol{\Omega}\boldsymbol{\Gamma} + (\boldsymbol{\Omega}\boldsymbol{\Gamma})^\top + \mathbf{U}_{ij}\frac{\partial\lambda_i}{\partial\xi_j} + \mu_i\mathbf{E}_i + \qquad (5.4\text{d})$$

$$\lambda_i\mathbf{K}_i + \mathbf{W}^\top \delta\mathbf{MW} = F_4.$$

$$(2\pi)^{1-n}\int_{\mathbb{T}^{n-1}} \left((\mathbf{W}\boldsymbol{\lambda})_1 + \boldsymbol{\chi} \cdot \nabla w_1 \right) d\boldsymbol{\xi} = f_5, \qquad (5.4\text{e})$$

$$(2\pi)^{1-n}\int_{\mathbb{T}^{n-1}} \left[\psi_0 - \delta\boldsymbol{\beta} \cdot \beta\mathbf{u} + w_2(\mathbf{W}\boldsymbol{\lambda})_1 \right] d\xi = 0 \qquad (5.4\text{f})$$

$$(2\pi)^{1-n}\int_{\mathbb{T}^{n-1}} \mathbf{V}^{-\top}\boldsymbol{\chi}\, d\xi = 0, \quad (2\pi)^{1-n}\int_{\mathbb{T}^{n-1}} (\mathbf{W}\boldsymbol{\Gamma})_{12}\, d\xi = 0, \qquad (5.4\text{g})$$

where $\delta\mathbf{m} = (\delta m, 0)^\top$ and $\delta\mathbf{M} = \text{diag}(\delta M, 0)$.

The difficulty is that the "angle variable" w_1 is a growing function of α. Hence the extra term $mw_1 + 2^{-1}Mw_1^2$ in the modified Hamiltonian has a polynomial growth in α. In other words, this means that the system of equations (5.4) contain secular terms. To cope with this difficulty we collect all secular terms together and introduce the new parameters

$$q = \delta e + \delta\boldsymbol{\beta}^\top \cdot \boldsymbol{\omega} + \alpha\delta m + 2^{-1}\alpha^2\delta M,$$
$$p = \delta m + \alpha\delta M, \quad \delta\mathbf{p} = (\delta p, 0) \qquad (5.5)$$

Recall that $\delta\boldsymbol{m} = (\delta m, 0)$, $\delta\boldsymbol{M} = \text{diag}\,(\delta M, 0)$. Thus we arrive at the following equations

$$\boldsymbol{\partial}\psi_0 + q + p \cdot (w_1 - \alpha) + \frac{1}{2}\delta M(w_1 - \alpha)^2 = F_1, \qquad (5.6\text{a})$$

$$\mathbf{J}\boldsymbol{\partial}\boldsymbol{\lambda} + \boldsymbol{\Omega}\boldsymbol{\lambda} + \mathbf{T}\boldsymbol{\mu} + p\mathbf{W}^\top\mathbf{e}_1 + \delta M(w_1 - \alpha)\mathbf{W}^\top\mathbf{e}_1 = F_2, \qquad (5.6\text{b})$$

$$\boldsymbol{\mu} = \nabla\psi_0 + \delta\boldsymbol{\beta}$$

$$-\boldsymbol{\partial}\boldsymbol{\chi} + \mathbf{S}\boldsymbol{\mu} + \mathbf{T}^\top\boldsymbol{\lambda} = F_3, \qquad (5.6\text{c})$$

$$\boldsymbol{\partial}(\mathbf{J}\boldsymbol{\Gamma}) + \boldsymbol{\Omega}\boldsymbol{\Gamma} + (\boldsymbol{\Omega}\boldsymbol{\Gamma})^\top + \mathbf{U}_{ij}\frac{\partial\lambda_i}{\partial\xi_j} + \mu_i\mathbf{E}_i + \lambda_i\mathbf{K}_i + \qquad (5.6\text{d})$$

$$+\mathbf{W}^\top \delta\mathbf{MW} = F_4, \quad \Gamma_{11} = -\Gamma_{22},$$

$$(2\pi)^{1-n} \int_{\mathbb{T}^{n-1}} \left((\mathbf{W}\boldsymbol{\lambda})_1 + \boldsymbol{\chi} \cdot \nabla w_1 \right) d\boldsymbol{\xi} = f_5, \tag{5.6e}$$

$$(2\pi)^{1-n} \int_{\mathbb{T}^{n-1}} \left[\psi_0 - \delta\boldsymbol{\beta} \cdot \mathbf{u} + w_2 \left(\mathbf{W}\boldsymbol{\lambda} \right)_1 \right] d\xi = 0 \tag{5.6f}$$

$$(2\pi)^{1-n} \int_{\mathbb{T}^{n-1}} \mathbf{V}^{-\top} \boldsymbol{\chi} \, d\xi = 0, \quad (2\pi)^{1-n} \int_{\mathbb{T}^{n-1}} (\mathbf{W}\boldsymbol{\Gamma})_{12} \, d\xi = 0. \tag{5.6g}$$

The following theorem constitutes the existence and uniqueness of solutions to problem (5.6)

Theorem 5.1. *Let a fixed $\sigma \in [1/4, 1]$, $d \geq 2$, and the matrix $\mathbf{K}_0 = \mathbf{S}_0 - \mathbf{t}_0 \otimes \mathbf{t}_0$ given by (2.11), satisfies the condition $\det \mathbf{K}_0 \neq 0$. Then there are $\varepsilon_0 > 0$ and r_0 with the following properties. For every*

$$(\alpha, k) \in \Sigma_\varrho, \quad \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_0(\alpha)\|_{\sigma, d} \leq r_0, \quad \boldsymbol{\varphi}_0(\alpha) = (0, 0, 0, \alpha \mathbf{e}_1, 1, 0, 0), \quad |\varepsilon| \leq \varepsilon_0, \quad ,$$

$$0 \leq \sigma_0 < \sigma_1 \leq \sigma, \quad \sigma_1 \geq 1/4,$$

and all $\mathbf{F} = (F_1, F_2, F_3, F_4, f_5) \in \mathcal{F}_{\sigma_1, 0}$, problem (5.6) has a unique solution

$$(\psi_0, \boldsymbol{\lambda}, \boldsymbol{\chi}, \boldsymbol{\Gamma}, \delta\boldsymbol{\beta}, q, p, \delta M) \in \mathcal{A}_{\sigma_0, 0} \times \mathcal{A}_{\sigma_0, 0}^2 \times \mathcal{A}_{\sigma, d}^{n-1} \times \mathcal{A}_{\sigma_0, 0}^4 \times \mathbb{C}^{n-1} \times \mathbb{C}^3.$$

This solution admits the estimate

$$\|(\psi_0, \boldsymbol{\lambda}, \boldsymbol{\chi}, \boldsymbol{\Gamma})\|_{\sigma_0, 0} + |(\delta\boldsymbol{\beta}, q, \ p, \delta M)| \leq c(\sigma_1 - \sigma_0)^{-8n-12} \|\mathbf{F}\|_{\sigma_1, 0} \tag{5.7}$$

where the constant c is independent of ε_0, r_0, and σ_i.

Proof. The proof is in Appendix E.1. □

Notice that problems (5.4) and (5.6) are equivalent. The vector $\boldsymbol{\Upsilon} = (\delta\boldsymbol{\beta}, \psi_0, \boldsymbol{\lambda}, \boldsymbol{\chi}, \boldsymbol{\Gamma})$ and the parameters δe, δm, and δM satisfy equations (5.4) if and only if $\boldsymbol{\Upsilon}$, δM and the parameters δq, δp, given by (5.5), satisfy equations (5.6). Thus we obtain the following

Corollary 5.2. *Under the assumptions of Theorem 5.1, problem (5.4) has a unique solution*

$$(\psi_0, \boldsymbol{\lambda}, \boldsymbol{\chi}, \boldsymbol{\Gamma}, \delta\boldsymbol{\beta}, \delta e, \delta m, \delta M) \in \mathcal{A}_{\sigma_0, 0} \times \mathcal{A}_{\sigma_0, 0}^2 \times \mathcal{A}_{\sigma, d}^{n-1} \times \mathcal{A}_{\sigma_0, 0}^4 \times \mathbb{C}^{n-1} \times \mathbb{C}^3.$$

This solution admits the estimate

$$\|(\psi_0, \boldsymbol{\lambda}, \boldsymbol{\chi}, \boldsymbol{\Gamma})\|_{\sigma_0, 0} + |(\delta\boldsymbol{\beta}, \delta M)| \leq c(\sigma_1 - \sigma_0)^{-8n-12} \|\mathbf{F}\|_{\sigma_1, 0} \tag{5.8}$$

$$|(\delta e, \delta m)| \leq (1 + \alpha^2) c(\sigma_1 - \sigma_0)^{-8n-12} \|\mathbf{F}\|_{\sigma_1, 0} \tag{5.9}$$

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5.2 Differential of Φ. Approximate inverse

In this section we construct an approximate inverse to the operator $D\Phi$. Our considerations are based on the following construction. For given φ denote by $\Xi(\varphi)$ the linear operator defined by the equalities

$$\Xi : \Upsilon = (\delta\boldsymbol{\beta}, \boldsymbol{\psi}_0, \boldsymbol{\lambda}, \boldsymbol{\chi}, \boldsymbol{\Gamma}) \to \delta\boldsymbol{\varphi} = (\delta\boldsymbol{\beta}, \delta\varphi_0, \delta\mathbf{u}, \delta\mathbf{w}, \delta W_{11}, \delta W_{12}, \delta W_{21}) \tag{5.10a}$$

$$\delta\mathbf{u} = \chi_i \frac{\partial}{\partial \xi_i}(\boldsymbol{\xi} + \mathbf{u}), \tag{5.10b}$$

$$\delta\mathbf{w} = \mathbf{W}\boldsymbol{\lambda} + \chi_i \frac{\partial}{\partial \xi_i}\mathbf{w}, \tag{5.10c}$$

$$\delta\mathbf{W} = \mathbf{W}\boldsymbol{\Gamma} + \chi_i \frac{\partial}{\partial \xi_i}\mathbf{W}, \tag{5.10d}$$

$$\delta\varphi_0 = \psi_0 + w_2\delta w_1 + \left(\frac{\partial \varphi_0}{\partial \xi_i} - w_2 \frac{\partial w_1}{\partial \xi_i}\right)\chi_i - \mathbf{u}\cdot\delta\boldsymbol{\beta}. \tag{5.10e}$$

In view of the Second structural Theorem 3.2 the operator Ξ has the inverse Ξ^{-1} defined by

$$\Xi^{-1}\delta\boldsymbol{\varphi} = (\delta\boldsymbol{\beta}, \delta\varphi_0, \delta\mathbf{u}, \delta\mathbf{w}, \delta W_{11}, \delta W_{12}, \delta W_{21})\Upsilon = (\delta\boldsymbol{\beta}, \boldsymbol{\psi}_0, \boldsymbol{\lambda}, \boldsymbol{\chi}, \boldsymbol{\Gamma}) \tag{5.11a}$$

$$\boldsymbol{\chi} = \mathbf{V}^\top \delta\mathbf{u} \tag{5.11b}$$

$$\boldsymbol{\lambda} = \mathbf{W}^{-1}\delta\mathbf{w} - \chi_i \mathbf{W}^{-1}\frac{\partial}{\partial \xi_i}\mathbf{w} \tag{5.11c}$$

$$\boldsymbol{\Gamma} = \mathbf{W}^{-1}\delta\mathbf{W} - \chi_i \mathbf{W}^{-1}\frac{\partial}{\partial \xi_i}\mathbf{W}, \tag{5.11d}$$

$$\boldsymbol{\mu} = \mathbf{V}^{-1}\left(\delta\mathbf{v} + \chi_i \frac{\partial}{\partial \xi_i}\mathbf{v} - \boldsymbol{\Lambda}\boldsymbol{\lambda}\right) \tag{5.11e}$$

$$\psi_0 = \delta\varphi_0 - \chi_i\frac{\partial \varphi_0}{\partial \xi_i} + w_2\chi_i\frac{\partial w_1}{\partial \xi_i} + \delta\boldsymbol{\beta}^\top \cdot \mathbf{u}. \tag{5.11f}$$

Recall the notation

$$\mathfrak{f} = (\alpha, k), \quad \mathfrak{u} = (\boldsymbol{\varphi}, e, m, M)$$

Introduce also the notation

$$\delta\mathfrak{u} = (\delta\boldsymbol{\varphi}, \delta e, \delta m, \delta M).$$

Definition 5.3. *Let $\mathcal{R}(\mathfrak{f}, \mathfrak{u})$ be a linear operator defined by the equality*

$$\mathcal{R}(\mathfrak{f}, \mathfrak{u})[\mathbf{F}] = \big(\boldsymbol{\Xi}(\boldsymbol{\varphi})[\boldsymbol{\Upsilon}], \delta e, \delta m, \delta M\big) \tag{5.12}$$

where $\boldsymbol{\Upsilon}$, δe, δm, δM is a solution to the approximate problem (5.4). Notice that the coefficients of equations (5.4) are completely defined by $\mathfrak{f}$ and $\mathfrak{u}$.

The following proposition constitutes the basic properties of the operator $\mathcal{R}$. In particular, it follows that $\mathcal{R}$ is an approximate inverse to the operator $D\boldsymbol{\Phi}$.

Proposition 5.4. *Let $\sigma \in (0, 1]$. Then there are $\varepsilon_0 > 0$ and $r_0 > 0$ with the following properties. For all $\mathfrak{f}$, $\mathfrak{u}$, and ε, and σ_i satisfying the conditions*

$$\mathfrak{f} \in \Sigma_\varrho, |\varepsilon| \le \varepsilon_0, \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_0(\alpha)\|_{\sigma,d} \le r_0,$$

$$0 \le \sigma_0 < \sigma_1 \le \sigma,$$

the operator $\mathcal{R}(\mathfrak{f}, \mathfrak{u}) : \mathcal{F}_{\sigma_1,0} \to \mathcal{E}_{\sigma_0,d}$ is bounded and admits the estimate

$$\|\mathcal{R}(\mathfrak{f}, \mathfrak{u})[\delta\mathfrak{u}]\|_{\sigma_0,d} \le c(1 + \alpha^2)(\sigma_1 - \sigma_0)^{-8n-12-d}\|\delta\mathfrak{u}\|_{\mathcal{E}_{\sigma_1,d}}. \tag{5.13}$$

Moreover, it satisfies the inequalities

$$\left\| \big(\mathcal{R}(\mathfrak{f}, \mathfrak{u})D\boldsymbol{\Phi}(\mathfrak{f}, \mathfrak{u}) - \mathrm{Id}\big)[\delta\mathfrak{u}] \right\|_{\sigma_0,d} \le$$
$$c(\sigma_1 - \sigma_0)^{-8n-13-2d}\|\boldsymbol{\Phi}(\mathfrak{f}, \mathfrak{u})\|_{\sigma,0}\|\delta\mathfrak{u}\|_{\mathcal{E}_{\sigma_1,d}}. \tag{5.14}$$

$$\left\| \big(D\boldsymbol{\Phi}(\mathfrak{f}, \mathfrak{u})\mathcal{R}(\mathfrak{f}, \mathfrak{u}) - \mathrm{Id}\big)[\mathbf{F}] \right\|_{\sigma_0,0} \le$$
$$c(\sigma_1 - \sigma_0)^{-8n-14}\|\boldsymbol{\Phi}(\mathfrak{f}, \mathfrak{u}))\|_{\sigma_1,0}\|\mathbf{F}\|_{\sigma_1,0}, \tag{5.15}$$

i.e., $\mathcal{R}$ is an approximate inverse to $D\boldsymbol{\Phi}$.

Proof. The proof is based on the following auxiliary lemma .

Lemma 5.5. *Let a fixed $\sigma \in (0, 1)$ and $d \ge 2$ and $r_0 < 1/2$. Then for every*

$$\mathfrak{f} \in \Sigma_\varrho, \quad \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_0(\alpha)\|_{\sigma,d} \le r_0, \quad \boldsymbol{\varphi}_0(\alpha) = (0, 0, 0, \boldsymbol{\alpha}, 1, 0, 0),$$

and every

$$0 \le \sigma_0 < \sigma_1 \le \sigma,$$

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the operator

$$\mathbf{\Xi} : \mathbb{C} \times \mathcal{A}_{\sigma_1,0} \times \mathcal{A}_{\sigma_1,0}^2 \times \mathcal{A}_{\sigma_1,0}^{n-1} \times \mathcal{A}_{\sigma_1,0}^4 \to \mathbb{C} \times \mathcal{A}_{\sigma_0,d} \times \mathcal{A}_{\sigma_0,d}^2 \times \mathcal{A}_{\sigma_1,d}^{n-1} \times \mathcal{A}_{\sigma_0,d}^3$$

is well defined and admits the estimate

$$\left\| \mathbf{\Xi}(\boldsymbol{\varphi}) \left[\mathbf{\Upsilon} \right] \right\|_{\sigma_0,d} \le c(\sigma_1 - \sigma_0)^{-d} \left\| \mathbf{\Upsilon} \right\|_{\sigma_1,0}. \tag{5.16}$$

The inverse operator

$$\mathbf{\Xi}^{-1} : \mathbb{C} \times \mathcal{A}_{\sigma_1,0} \times \mathcal{A}_{\sigma_1,0}^2 \times \mathcal{A}_{\sigma_1,0}^{n-1} \times \mathcal{A}_{\sigma_1,0}^4 \to \mathbb{C} \times \mathcal{A}_{\sigma_0,d} \times \mathcal{A}_{\sigma_0,d}^2 \times \mathcal{A}_{\sigma_1,d}^{n-1} \times \mathcal{A}_{\sigma_0,d}^3$$

is well defined and admits the estimate

$$\left\| \mathbf{\Xi}^{-1}(\boldsymbol{\varphi}) \left[\delta\boldsymbol{\varphi} \right] \right\|_{\sigma_0,d} \le c(\sigma_1 - \sigma_0)^{-d} \left\| \delta\boldsymbol{\varphi} \right\|_{\sigma_1,0}. \tag{5.17}$$

Proof. In order to avoid repetitions we prove estimate (5.16). The proof of estimate (5.17) is similar. Notice that for $d \ge 1$ and $r_0 \le 1/2$, the equality $W_{22} = W_{11}^{-1}(1 - W_{1,2}W_{21})$ implies

$$\|w_2\|_{\sigma,0} + \|\nabla w_1\|_{\sigma,0} + \|\nabla \mathbf{W}\|_{\sigma,0} + \|\mathbf{W} - \mathbf{I}\|_{\sigma,0} \le c\|\boldsymbol{\varphi} - \boldsymbol{\varphi}(\alpha)\|_{\sigma,d}$$

Obviously we have

$$\|w_2\|_{\sigma,0} + \|\nabla w_1\|_{\sigma,0} + \|\nabla(\boldsymbol{\xi} + \boldsymbol{u})\| + \|\mathbf{W}\|_{\sigma,0} \le c\|\boldsymbol{\varphi} - \boldsymbol{\varphi}(\alpha)\|_{\sigma,d} \le c$$

Since $\mathcal{A}_{\sigma_1,0}$ is a Banach algebra, it follows from this and (5.10) that

$$\|\mathbf{\Xi}(\boldsymbol{\varphi})[\mathbf{\Upsilon}]\|_{\sigma_1,0} \le c\|\mathbf{\Upsilon}]\|_{\sigma_1,0}.$$

It remains to note that

$$\|\mathbf{\Xi}(\boldsymbol{\varphi})[\mathbf{\Upsilon}]\|_{\sigma_0,d} \le c(\sigma_1 - \sigma_0)^{-d}\|\mathbf{\Xi}(\boldsymbol{\varphi})[\mathbf{\Upsilon}]\|_{\sigma_1,0}.$$

$\square$

Let us turn to the proof of the proposition. The proof falls into three steps.

Step 1. We begin with proving of estimate (5.13). It follows from Corollary 5.2 that, under the assumptions of Proposition 5.4, for a suitable choice

of ε_0 and r_0, problem (5.4) has the unique solution $\Upsilon = (\delta\boldsymbol{\beta}, \psi_0, \boldsymbol{\lambda}, \boldsymbol{\chi}, \boldsymbol{\Gamma})$, and δe, δm, δM satisfying the inequality

$$\|\Upsilon\|_{(\sigma_0+\sigma_1)/2,0} + |\delta e| + |\delta m| + |\delta M|$$
$$\leq c(1+\alpha^2)(\sigma_1 - \sigma_0)^{-8n-12} \|\mathbf{F}\|_{\sigma_1,0}. \quad (5.18)$$

Using this estimate and applying Lemma 5.5 with σ_1 replaced by $(\sigma_0 + \sigma_1)/2$ we obtain

$$\left\| \Xi(\boldsymbol{\varphi})\,[\Upsilon] \right\|_{\sigma_0,d} \leq c\,(\sigma_1 - \sigma_0)^{-d} \|\Upsilon\|_{(\sigma_0+\sigma_1)/2,0}$$
$$\leq c(\sigma_1 - \sigma_0)^{-8n-12-d} \|\mathbf{F}\|_{\sigma_1,0}. \quad (5.19)$$

Combining (5.18) and (5.19) and recalling Definition 5.3 we arrive at (5.13).

Step 2. Now our task is to prove estimate (**??**). Assume that ε, r_0 meet all requirements of Corollary 5.2. Choose an arbitrary

$$\delta\boldsymbol{\varphi} = (\delta\boldsymbol{\beta}, \delta\varphi_0, \delta\mathbf{u}, \delta W_{11}, \delta W_{12}, \delta W_{21}) \quad \text{and} \quad (\delta e, \delta m, \delta M)$$

such that
$$\delta\mathfrak{u} \equiv (\delta\boldsymbol{\varphi}, \delta e, \delta m, \delta M) \in \mathcal{E}_{\sigma_1,d}.$$

Set
$$\Upsilon = \Xi^{-1}(\boldsymbol{\varphi})[\delta\boldsymbol{\varphi}] \quad (5.20)$$

By construction, the vector

$$\Upsilon = (\delta\boldsymbol{\beta}, \psi_0, \boldsymbol{\lambda}, \boldsymbol{\chi}, \boldsymbol{\Gamma})$$

is connected with $\delta\boldsymbol{\varphi}$ by relations (4.22). Hence Υ and the parameters $(\delta e, \delta m, \delta M)$ meet all requirements of Corollary 4.8. It follows that they satisfy identities (4.30). Notice that relations (4.30) can be regarded as system of equations (5.4a)-(5.4e) with the right hand sides

$$F_1 = -\Pi_1[\boldsymbol{\chi}, \boldsymbol{\lambda}, \boldsymbol{\mu}] + D_{\mathfrak{u}}\Phi_1(\mathfrak{f}, \mathfrak{u})[\delta\mathfrak{u}],$$
$$F_2 = -\Pi_2[\boldsymbol{\chi}, \boldsymbol{\lambda}, \boldsymbol{\Gamma}] + D_{\mathfrak{u}}\Phi_2(\mathfrak{f}, \mathfrak{u})[\delta\mathfrak{u}],$$
$$F_3 = -\Pi_3[\boldsymbol{\chi}] + D_{\mathfrak{u}}\Phi_3(\mathfrak{f}, \mathfrak{u})[\delta\mathfrak{u}], \quad (5.21)$$
$$F_4 = -\Pi_2[\boldsymbol{\chi}, \boldsymbol{\Gamma}] + D_{\mathfrak{u}}\Phi_4(\mathfrak{f}, \mathfrak{u})[\delta\mathfrak{u}],$$
$$f_5 = D_{\mathfrak{u}}\Phi_5(\mathfrak{f}, \mathfrak{u})[\delta\mathfrak{u}]$$

Moreover, since $\delta\mathfrak{u} \in \mathcal{E}_{\sigma_1,d}$, its component $\delta\boldsymbol{\varphi}$ satisfies the integral relations (5.2) which, in view of (5.10b)-(5.10e), are equivalent to the integral conditions (5.4f)-(5.4g). Therefore, the vector $\boldsymbol{\Upsilon}$ and the parameters $(\delta e, \delta m, \delta M)$ satisfy equations (5.4) with the right hand sides given by (5.21).

On the other hand, Definition 5.3 implies the identity

$$\mathcal{R}(\mathfrak{f}, \mathfrak{u}) \left[D_\mathfrak{u}\boldsymbol{\Phi}(\mathfrak{f}, \mathfrak{u})[\delta\mathfrak{u}] \right] = \left(\boldsymbol{\Xi}(\boldsymbol{\varphi})[\boldsymbol{\Upsilon}'], \delta e', \delta m', \delta M' \right) \qquad (5.22)$$

where the vector $\boldsymbol{\Upsilon}'$ and the parameters $(\delta e', \delta m', \delta M')$ satisfy equations (5.4) with the right hand sides

$$F_i' = D\Phi_i(\mathfrak{f}, \mathfrak{u})[\delta\mathfrak{u}], \ i = 1, \ldots, 4, \ f_5' = D\Phi_5(\mathfrak{f}, \mathfrak{u})[\delta\mathfrak{u}] \qquad (5.23)$$

It follows from (5.20) that

$$\mathcal{R}(\mathfrak{f}, \mathfrak{u}) \left[D_\mathfrak{u}\boldsymbol{\Phi}(\mathfrak{f}, \mathfrak{u})[\delta\mathfrak{u}] \right] - \delta\mathfrak{u} = \left(\boldsymbol{\Xi}(\boldsymbol{\varphi})[\tilde{\boldsymbol{\Upsilon}}], \tilde{\delta e}, \tilde{\delta m}, \tilde{\delta M} \right) \qquad (5.24)$$

Where

$$\tilde{\boldsymbol{\Upsilon}} = \boldsymbol{\Upsilon}' - \boldsymbol{\Upsilon}, \quad \tilde{\delta e} = \delta e' - \delta e, \quad \tilde{\delta m} = \delta m' - \delta m, \quad \tilde{\delta M} = \delta M' - \delta M.$$

Notice that $(\boldsymbol{\Upsilon}, \delta e, \delta m, \delta M)$ satisfies equations (5.4) with the right hand sides (5.21). On the other hand, $(\boldsymbol{\Upsilon}', \delta e', \delta m', \delta M')$ satisfies equations (5.4) with the right hand sides (5.23). It follows from this that $(\tilde{\boldsymbol{\Upsilon}}, \tilde{\delta e}, \tilde{\delta m}, \tilde{\delta M})$ satisfies equations (5.4) with the right hand sides

$$\tilde{F}_i = \Pi_i[\boldsymbol{\Upsilon}], \ i = 1, \ldots, 4, \ \tilde{f}_5 = 0. \qquad (5.25)$$

Here the operators Π_i are given by formulae (4.24):

$$\Pi_1[\boldsymbol{\Upsilon}] = \frac{\partial\Phi_1}{\partial\xi_i}\chi_i + \Phi_2^\top \cdot \boldsymbol{\lambda} + \Phi_3^\top \cdot \boldsymbol{\mu}, \qquad (5.26a)$$

$$\Pi_2[\boldsymbol{\chi}, \boldsymbol{\lambda}, \boldsymbol{\Gamma}] = \chi_i\frac{\partial\Phi_2}{\partial\xi_i} + \boldsymbol{\Gamma}^\top\Phi_2 + \Phi_4\boldsymbol{\lambda} + (\mathbf{J}\boldsymbol{\lambda})_\xi'\Phi_3, \qquad (5.26b)$$

$$\Pi_3[\boldsymbol{\chi}] = \chi_i\frac{\partial\Phi_3}{\partial\xi_i} - \boldsymbol{\chi}_\xi'\Phi_3, \qquad (5.26c)$$

$$\Pi_4[\boldsymbol{\chi}, \boldsymbol{\Gamma}] = \chi_i\frac{\partial\Phi_4}{\partial\xi_i} + \Phi_{3,i}\frac{\partial}{\partial\xi_i}(\mathbf{J}\boldsymbol{\Gamma}) + \Phi_4\boldsymbol{\Gamma} + (\Phi_4\boldsymbol{\Gamma})^\top, \qquad (5.26d)$$

where $\Phi_i = \Phi_i(\mathfrak{f}, \mathfrak{u})$ and $\boldsymbol{\mu} = \delta\boldsymbol{\beta} + \nabla\psi_0$. We thus get

$$\boldsymbol{\partial}\tilde{\psi}_0 + \boldsymbol{\omega}^\top \cdot \delta\tilde{\boldsymbol{\beta}} + \tilde{\delta e} + \delta\tilde{\mathbf{m}} \cdot \mathbf{w} + \frac{1}{2}\mathbf{w}^\top \delta\tilde{\mathbf{M}}\mathbf{w} = \tilde{F}_1, \tag{5.27a}$$

$$\mathbf{J}\boldsymbol{\partial}\tilde{\boldsymbol{\lambda}} + \boldsymbol{\Omega}\tilde{\boldsymbol{\lambda}} + \mathbf{T}\tilde{\boldsymbol{\mu}} + \tilde{\delta m}\mathbf{W}^\top\mathbf{e}_1 + \mathbf{W}^\top\delta\tilde{\mathbf{M}}\mathbf{w} = \tilde{F}_2, \tag{5.27b}$$

$$\tilde{\boldsymbol{\mu}} = \nabla\tilde{\psi}_0 + \delta\tilde{\boldsymbol{\beta}}$$

$$-\boldsymbol{\partial}\tilde{\boldsymbol{\chi}} + \mathbf{S}\tilde{\boldsymbol{\mu}} + \mathbf{T}^\top\tilde{\boldsymbol{\lambda}} = \tilde{F}_3, \tag{5.27c}$$

$$\boldsymbol{\partial}(\mathbf{J}\tilde{\boldsymbol{\Gamma}}) + \boldsymbol{\Omega}\tilde{\boldsymbol{\Gamma}} + (\boldsymbol{\Omega}\tilde{\boldsymbol{\Gamma}})^\top + \mathbf{U}_{ij}\frac{\partial\lambda_i}{\partial\xi_j^*} + \tilde{\mu}_i\mathbf{E}_i+ \tag{5.27d}$$

$$\tilde{\lambda}_i\mathbf{K}_i + \mathbf{W}^\top\delta\tilde{\mathbf{M}}\mathbf{W} = \tilde{F}_4.$$

$$\int_{\mathbb{T}^{n-1}} \left((\mathbf{W}\tilde{\boldsymbol{\lambda}})_1 + \tilde{\boldsymbol{\chi}} \cdot \nabla w_1\right) d\boldsymbol{\xi} = 0, \tag{5.27e}$$

$$\int_{\mathbb{T}^{n-1}} \left[\tilde{\psi}_0 - \delta\tilde{\boldsymbol{\beta}} \cdot \mathbf{u} + w_2(\mathbf{W}\tilde{\boldsymbol{\lambda}})_1\right] d\xi = 0 \tag{5.27f}$$

$$\int_{\mathbb{T}^{n-1}} \mathbf{V}^{-\top}\tilde{\boldsymbol{\chi}} \, d\xi = 0, \quad \int_{\mathbb{T}^{n-1}} (\mathbf{W}\tilde{\boldsymbol{\Gamma}})_{12} \, d\xi = 0, \tag{5.27g}$$

where $\delta\tilde{\mathbf{M}} = \text{diag}(\delta\tilde{M}, 0)$. Let us estimate $\tilde{\mathbf{F}}$. Equality $\boldsymbol{\Upsilon} = \boldsymbol{\Xi}^{-1}\boldsymbol{\delta\varphi}$ and estimate (5.17) imply

$$\|\boldsymbol{\Upsilon}\|_{(\sigma_1+\sigma_0)/2,d} \leq c(\sigma_1 - \sigma_0)^{-d} \|\delta\boldsymbol{\varphi}\|_{\sigma_1,0}$$

It follows from this that

$$\|\boldsymbol{\Upsilon}\|_{(2\sigma_1+\sigma_0)/3,0} + \|\partial_\xi\boldsymbol{\Upsilon}\|_{(2\sigma_1+\sigma_0)/3,0} \leq c(\sigma_1 - \sigma_0)^{-d}\|\delta\boldsymbol{\varphi}\|_{\sigma_1,0}. \tag{5.28}$$

Next we have

$$\|\boldsymbol{\Phi}\|_{(2\sigma_1+\sigma_0)/3,0} + \|\partial_\xi\boldsymbol{\Phi}\|_{(2\sigma_1+\sigma_0)/3,0} \leq c(\sigma_1 - \sigma_0)^{-1}\|\boldsymbol{\Phi}\|_{\sigma_1,0}. \tag{5.29}$$

Combining estimates (5.28) and (5.29) and recalling formulae (5.26) we arrive at

$$\|\Pi[\boldsymbol{\Upsilon}]\|_{(2\sigma_1+\sigma_0)/3,0} \leq c(\sigma_1 - \sigma_0)^{-d-1}\|\boldsymbol{\Phi}\|_{\sigma_1,0} \, \|\delta\boldsymbol{\varphi}\|_{\sigma_1,0} \tag{5.30}$$

Since $\delta\boldsymbol{\varphi}$ is the component of the vector $\mathfrak{u}$, it follows from this and (5.25) that

$$\|\tilde{\mathbf{F}}\|_{(2\sigma_1+\sigma_0)/3,0} \leq c(\sigma_1 - \sigma_0)^{-d-1} \|\boldsymbol{\Phi}\|_{\sigma_1,0} \, \|\delta\mathfrak{u}\|_{\sigma_1,0} \tag{5.31}$$

Applying Corollary 5.2 to problem (5.27) we conclude that it has the unique solution. Moreover, estimates (5.8) and (5.9) in this corollary imply the estimate

$$c(1 + |\alpha^2|)(\sigma_1 - \sigma_0)^{-8n-12} \|\tilde{\mathbf{F}}\|_{(2\sigma_1 + \sigma_0)/3, 0}.$$

This result along with inequality (5.31) leads to the estimate

$$\|\tilde{\mathbf{\Upsilon}}\|_{(\sigma_1 + \sigma_0)/2, 0} + |\delta\tilde{e}| + |\delta\tilde{m}| + |\delta\tilde{M}| \leq$$
$$c(1 + |\alpha^2|)(\sigma_1 - \sigma_0)^{-8n-13-d} \|\mathbf{\Phi}\|_{\sigma_1, 0} \|\delta\mathfrak{u}\|_{\sigma_1, 0}.$$

From this, relation (5.24) and estimate (5.16) for the norm of the operator Ξ we obtain

$$\left\| \mathcal{R}(\mathfrak{f}, \mathfrak{u}) \left[D_{\mathfrak{u}} \mathbf{\Phi}(\mathfrak{f}, \mathfrak{u}) [\delta\mathfrak{u}] \right] - \delta\mathfrak{u} \right\|_{\sigma_0, d} =$$
$$\left\| \Xi(\boldsymbol{\varphi})[\tilde{\mathbf{\Upsilon}}] \right\|_{\sigma_0, d} + |\delta\tilde{e}| + |\delta\tilde{m}| + |\delta\tilde{M}| \leq$$
$$(\sigma_1 - \sigma_0)^{-d} \|\tilde{\mathbf{\Upsilon}}\|_{(\sigma_1 + \sigma_0)/2, 0} + |\delta\tilde{e}| + |\delta\tilde{m}| + |\delta\tilde{M}| \leq$$
$$c(1 + |\alpha^2|)(\sigma_1 - \sigma_0)^{-8n-13-2d} \|\mathbf{\Phi}\|_{\sigma_1, 0} \|\delta\mathfrak{u}\|_{\sigma_1, 0}.$$

which obviously leads to (5.14).

Step 3. It remains to prove estimate (5.15). Choose an arbitrary $\mathbf{F} \in \mathcal{F}_{\sigma_1, 0}$ and set $\delta\mathfrak{u} := \mathcal{R}(\mathfrak{f}, \mathfrak{u})[\mathbf{F}]$. It follows from the definition 5.3 that

$$\delta\mathfrak{u} = (\Xi[\mathbf{\Upsilon}], \delta e, \delta m, \delta M),$$

where $(\mathbf{\Upsilon}, \delta e, \delta m, \delta M)$ is a solution to problem (5.4). On the other hand, in view of Corollary 4.8, the vector

$$D_{\mathfrak{u}} \mathbf{\Phi}(\mathfrak{f}, \mathfrak{u}) \mathcal{R}(\mathfrak{f}, \mathfrak{u})[\mathbf{F}] \equiv D_{\mathfrak{u}} \mathbf{\Phi}(\mathfrak{f}, \mathfrak{u}) \delta\mathfrak{u}$$

satisfies identities (4.30). Combining (4.30) and (5.4) we arrive at the identity

$$D_{\mathfrak{u}} \mathbf{\Phi}(\mathfrak{f}, \mathfrak{u}) \mathcal{R}(\mathfrak{f}, \mathfrak{u})[\mathbf{F}] - \mathbf{F} = (\Pi_1[\mathbf{\Upsilon}], \Pi_2[\mathbf{\Upsilon}], \Pi_3[\mathbf{\Upsilon}], \Pi_4[\mathbf{\Upsilon}], \Pi_5[\mathbf{\Upsilon}]). \quad (5.32)$$

Applying Corollary 5.2 to equations (5.4) we conclude that

$$\|\mathbf{\Upsilon}\|_{(\sigma_0 + \sigma_1)/2, 0} + |\delta e| + |\delta m| + |\delta M| \leq c(1 + |\alpha^2|)(\sigma_1 - \sigma_0)^{-8n-12} \|\mathbf{F}\|_{\sigma_1, 0},$$

which gives

$$\|\mathbf{\Upsilon}\|_{\sigma_0, 0} + \|\partial_\xi \mathbf{\Upsilon}\|_{\sigma_0, 0} \leq c(\sigma_1 - \sigma_0)^{-8n-13} \|\mathbf{F}\|_{\sigma_1, 0}. \quad (5.33)$$

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Next we have

$$\|\mathbf{\Phi}\|_{\sigma_0,0} + \|\partial_\xi \mathbf{\Phi}\|_{\sigma_0,0} \leq c(\sigma_1 - \sigma_0)^{-1}\|\mathbf{\Phi}\|_{\sigma_1,0}. \tag{5.34}$$

Combining estimates (5.33) and (5.34) and recalling formulae (5.26) we arrive at

$$\|\Pi[\Upsilon]\|_{\sigma_0,0} \leq c(\sigma_1 - \sigma_0)^{-8n-14}\|\Phi_i\|_{\sigma_1,0}\|\mathbf{F}\|_{\sigma_1,0}. \tag{5.35}$$

It remains to note that desired estimate (5.15) is a straightforward consequence of estimate (5.35) and equality (5.32). □

6 Implicit function theorem

6.1 Nash-Moser-Zehnder Implicit Function Theorem

We prove the local solvability of operator equation (4.16) by using the nash-Moser implicit function theorem. There are many different versions of this celebrated theorem. Our considerations are based on the version of the Nash-Moser implicit function theorem proposed by Zhender, see [15], [24].

Fix $d \geq 2$. Recall Definitions 4.4 and 4.5 of spaces $\mathcal{E}_{\sigma,d}$ and $\mathcal{F}_{\sigma,d}$. Let us consider the abstract operator equation

$$\mathbf{\Phi}(\mathfrak{f}, \mathfrak{u}) = 0, \quad \mathfrak{f} \in \mathbb{C}^2, \quad \mathfrak{u} \in \mathcal{E}_{\sigma,d}. \tag{6.1}$$

Here $\mathbf{\Phi}$ is a smooth operator. Assume that the operator $\mathbf{\Phi}$ satisfies the following conditions, cf. [15], ch.6.1.

For fixed $R > 0$, $N > 0$, and $(\mathfrak{f}_0, \mathfrak{u}_0) \in \mathbb{C}^2 \times \mathcal{E}_{1,d}$ denote by $\mathcal{B}_\sigma(\mathfrak{f}_0, \mathfrak{u}_0) \subset \mathbb{C}^2 \times \mathcal{E}_{\sigma,d}$ the ball

$$\mathcal{B}_\sigma(\mathfrak{f}_0, \mathfrak{u}_0) = \{(\mathfrak{f}, \mathfrak{u}) : \ |\mathfrak{f} - \mathfrak{f}_0| < N, \quad \|\mathfrak{u} - \mathfrak{u}_0\|_{\sigma,d} < R\}. \tag{6.2}$$

H.3 The mapping $\mathbf{\Phi}$ is defined in $\mathcal{B}_0(\mathfrak{f}_0, \mathfrak{u}_0)$. Moreover, the mapping $\mathbf{\Phi} : \mathcal{B}_\sigma(\mathfrak{f}_0, \mathfrak{u}_0) \to \mathcal{F}_{\sigma,0}$ is continuous for all $\sigma \in (0, 1]$. For every $\sigma' < \sigma$, the mapping $\mathbf{\Phi}(\mathfrak{f}, \cdot) : \mathcal{B}_\sigma(\mathfrak{f}_0, \mathfrak{u}_0) \cap \mathcal{E}_{\sigma,d} \to \mathcal{F}_{\sigma',0}$, $\sigma' < \sigma$ is differentiable. For every $(\mathfrak{f}, \mathfrak{u}) \in \mathcal{B}_\sigma(\mathfrak{f}_0, \mathfrak{u}_0)$, the quantity

$$Q(\mathfrak{f}; \mathfrak{u}, \mathfrak{v}) = \mathbf{\Phi}(\mathfrak{f}, \mathfrak{u}) - \mathbf{\Phi}(\mathfrak{f}, \mathfrak{v}) - D_\mathfrak{v}\mathbf{\Phi}(\mathfrak{f}, \mathfrak{v})(\mathfrak{u} - \mathfrak{v})$$

admits the estimate

$$\|Q(\mathfrak{f}; \mathfrak{u}, \mathfrak{v})\|_{\sigma',0} \leq c_0(\sigma - \sigma')^{-2\tau}\|\mathfrak{u} - \mathfrak{v}\|_{\sigma,d}^2. \tag{6.3}$$

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H.4 For every $\sigma \in (0, 1]$ and $(\mathfrak{f}, \mathfrak{u}), (\mathfrak{g}, \mathfrak{u}) \in \mathcal{B}_\sigma(\mathfrak{f}_0, \mathfrak{u}_0)$, we have

$$\|\mathbf{\Phi}(\mathfrak{f}, \mathfrak{u}) - \mathbf{\Phi}(\mathfrak{g}, \mathfrak{u})\|_{\sigma, 0} \le c_0 |\mathfrak{f} - \mathfrak{g}|. \tag{6.4}$$

H.5 For every $\sigma \in (0, 1]$, every $0 \le \sigma' < \sigma$, and every $(\mathfrak{f}, \mathfrak{u}), \in \mathcal{B}_\sigma(\mathfrak{f}_0, \mathfrak{u}_0)$, there exists the linear continuous mapping $\mathcal{R}(\mathfrak{f}, \mathfrak{u}) : \mathcal{F}_{\sigma, 0} \to \mathcal{E}_{\sigma', d}$ such that for all $\mathfrak{h} \in \mathcal{F}_{\sigma, 0}$ and all $\mathfrak{v} \in \mathcal{E}_{\sigma, d}$,

$$\|\mathcal{R}(\mathfrak{f}, \mathfrak{u})\mathfrak{h}\|_{\sigma', d} \le c_0 (\sigma - \sigma')^{-\gamma} \|\mathfrak{h}\|_{\sigma, 0} \tag{6.5}$$

$$\|\big(D_\mathfrak{u}\mathbf{\Phi}(\mathfrak{f}, \mathfrak{u})\, \mathcal{R}(\mathfrak{f}, \mathfrak{u}) - \mathbf{I}\big)\mathfrak{h}\|_{\sigma', 0}$$
$$\le c_0(\sigma - \sigma')^{-2\tau - \gamma} \|\, \mathbf{\Phi}(\mathfrak{f}, \mathfrak{u})\|_{\sigma, 0}\, \|\mathfrak{h}\|_{\sigma, 0}. \tag{6.6}$$

$$\|\big(\mathcal{R}(\mathfrak{f}, \mathfrak{u})\, D_\mathfrak{u}\mathbf{\Phi}(\mathfrak{f}, \mathfrak{u}) - \mathbf{I}\big)\mathfrak{v}\|_{\sigma', d}$$
$$\le c_0(\sigma - \sigma')^{-2\tau - \gamma} \|\, \mathbf{\Phi}(\mathfrak{f}, \mathfrak{u})\|_{\sigma, 0}\, \|\mathfrak{v}\|_{\sigma, d}. \tag{6.7}$$

The following theorem, see [15], Theorem 6.1 and Corollaries, constitute the local existence and uniqueness of solutions to operator equation (6.1)

Theorem 6.1. *Assume that* $\mathbf{\Phi}$ *satisfies Conditions* $(\mathbf{H.3}) - (\mathbf{H.5})$. *Then there exists a constant* C, *depending only on* c_0, τ, *and* γ *with the following properties. If for some* $\sigma \in (0, 1]$, *the couple* $(\mathfrak{f}, \mathfrak{v}(\mathfrak{f})) \in \mathcal{B}_\sigma(\mathfrak{f}_0, \mathfrak{u}_0)$ *satisfies the conditions*

$$\|\mathfrak{v}(\mathfrak{f}) - \mathfrak{u}_0\|_{\sigma, d} \le r \le R, \quad \|\mathbf{\Phi}(\mathfrak{f}, \mathfrak{v}(\mathfrak{f}))\|_{\sigma, 0} \le C(R - r)\sigma^{2(\tau + \gamma)}, \tag{6.8}$$

then equation (6.1) have a solution $\mathfrak{u} = \mathfrak{u}(\mathfrak{f}) \in \mathcal{E}_{\sigma/2, d}$ *such that*

$$\|\mathfrak{v}(\mathfrak{f}) - \mathfrak{u}(\mathfrak{f})\|_{\sigma/2, d} \le C^{-1}\|\mathbf{\Phi}(\mathfrak{f}, \mathfrak{v}(\mathfrak{f}))\|_{\sigma, 0}\sigma^{-\gamma}. \tag{6.9}$$

Moreover, if the mapping $\mathcal{B}_\sigma \cap \mathbb{C}^2 \ni \mathfrak{f} \to \mathfrak{v}(\mathfrak{f}) \in \mathcal{E}_{\sigma, d}$ *is continuous, then the mapping* $\mathcal{B}_\sigma(\mathfrak{f}_0, \mathfrak{u}_0) \cap \mathbb{C}^2 \ni \mathfrak{f} \to \mathfrak{u}(\mathfrak{f}) \in \mathcal{E}_{\sigma/2, d}$ *is continuous. If, in addition,* $\mathfrak{v}(\mathfrak{f})$ *satisfies the inequality*

$$2C^{-1}\|\mathbf{\Phi}(\mathfrak{f}, \mathfrak{v}(\mathfrak{f}))\|_{\sigma, 0}\sigma^{-\gamma} < 1, \tag{6.10}$$

then the solution $\mathfrak{u}(\mathfrak{f})$ *is unique.*

Proof. Existence of solution $\mathfrak{u}(\mathfrak{f})$ is exactly the statement Theorem 6.1 in [15]. The continuity of this solution is a consequence of the Corollary of this theorem, see [15]. The uniqueness also results from ch.6 in[15]. However, since our formulation is slightly different, we recall these arguments. Let $\mathfrak{u}(f), \mathfrak{u}'(\mathfrak{f}) \in \mathcal{B}_\sigma(\mathfrak{f}_0, \mathfrak{u}_0)$ satisfy equation (6.1) and inequalities (6.9). It follows that

$$D_{\mathfrak{u}}\Phi(\mathfrak{f}, \mathfrak{u}(\mathfrak{f}))(\mathfrak{u}'(\mathfrak{f}) - \mathfrak{u}(\mathfrak{f})) = -Q(\mathfrak{f}, \mathfrak{u}, \mathfrak{u}')$$

Applying to both sides of this identity the operator $\mathcal{R}(\mathfrak{f}, \mathfrak{u}(\mathfrak{f})$ and recalling that $\mathfrak{u}(\mathfrak{f})$ satisfies equation (6.1) we obtain

$$\mathfrak{u}'(\mathfrak{f}) - \mathfrak{u}(\mathfrak{f}) = -\mathcal{R}(\mathfrak{f}, \mathfrak{u}(\mathfrak{f})Q(\mathfrak{f}, \mathfrak{u}, \mathfrak{u}').$$

Choose an arbitrary $\sigma'' < \sigma' \le \sigma$.Applying inequalities (6.3) and (6.5) we arrive at

$$\|\mathfrak{u}'(\mathfrak{f}) - \mathfrak{u}(\mathfrak{f})\|_{\sigma'',d} \le c(\sigma' - \sigma'')^{-\gamma}\|Q(\mathfrak{f}, \mathfrak{u}, \mathfrak{u}')\|_{(\sigma'+\sigma'')/2,d} \le$$
$$c(\sigma' - \sigma'')^{-2\tau-\gamma}\|\mathfrak{u}'(\mathfrak{f}) - \mathfrak{u}(\mathfrak{f})\|_{\sigma'',d}^2.$$

Now set $\sigma_n = 2^{-n-1}\sigma$, $n \ge 0$. We have

$$\|\mathfrak{u}'(\mathfrak{f}) - \mathfrak{u}(\mathfrak{f})\|_{\sigma_{n+1},d} \le c2^{n(\gamma+2\tau)}\|\mathfrak{u}'(\mathfrak{f}) - \mathfrak{u}(\mathfrak{f})\|_{\sigma_n,d},$$

which leads to the inequality

$$\|\mathfrak{u}'(\mathfrak{f}) - \mathfrak{u}(\mathfrak{f})\|_{\sigma_{n+1},d} \le c^n 2^{n^2(\gamma+2\tau)}\|\mathfrak{u}'(\mathfrak{f}) - \mathfrak{u}(\mathfrak{f})\|_{\sigma/2,d}^{2^n},$$

On the other hand, estimate (6.9) imply

$$\|\mathfrak{u}'(\mathfrak{f}) - \mathfrak{u}(\mathfrak{f})\|_{\sigma/2,d} \le 2C^{-1}\|\Phi(\mathfrak{f}, \mathfrak{v}(\mathfrak{f}))\|_{\sigma,0}\sigma^{-\gamma}.$$

Combining the obtained results and recalling (6.10) we finally obtain

$$\|\mathfrak{u}'(\mathfrak{f}) - \mathfrak{u}(\mathfrak{f})\|_{0,d} \le \|\mathfrak{u}'(\mathfrak{f}) - \mathfrak{u}(\mathfrak{f})\|_{\sigma_{n+1},d} \le$$
$$c^n 2^{n^2(\gamma+2\tau)}(2C^{-1}\|\Phi(\mathfrak{f}, \mathfrak{v}(\mathfrak{f}))\|_{\sigma,0}\sigma^{-\gamma})^{2^n} \to 0 \text{ as } n \to \infty.$$

$\square$

6.2   Solvability of the main operator equation

We are now in a position to prove the local solvability and uniqueness result for the main operator equation (4.16). Recall the notation

$$\mathfrak{f} = (\alpha, k), \ \mathfrak{u} = (\boldsymbol{\varphi}, e, m, M) \text{ where } \boldsymbol{\varphi} = (\boldsymbol{\beta}, \varphi_0, \mathbf{u}, \mathbf{w}, W_{11}, W_{12}, W_{22}). \quad (6.11)$$

and

$$\mathfrak{v}(\mathfrak{f}) = (\boldsymbol{\varphi}_0(\alpha), \ -k\alpha - k\alpha^2/2, \ k\alpha, \ -k), \quad (6.12)$$

where

$$\boldsymbol{\varphi}_0(\alpha) = (0, 0, \alpha \mathbf{e}_1, 1, 0, 0).$$

Fix an arbitrary $\sigma \in (0, 1]$ and $d \geq 1$. Next, fix $\varrho > 0$ satisfying Condition (**H.1**) of Theorem 2.2 and set

$$G = \{(\alpha, k) \in \mathbb{C}^2 : \ |\operatorname{Re} \alpha| \leq 4\pi, \operatorname{Re} k \in [0, 1], \ |\operatorname{Im} \alpha| \leq \varrho, |\operatorname{Im} k| \leq \varrho\}. \quad (6.13)$$

The following theorem is the main result of this section

Theorem 6.2. *Let conditions (**H.1**)-(**H.2**) be satisfied. Then there is $\varepsilon_0 > 0$ such that for all $|\varepsilon| \leq \varepsilon_0$ and for all $\mathfrak{f} \in G$ operator equation (4.16) has a unique solution $\mathfrak{u}(\mathfrak{f}) \in \mathcal{E}_{\sigma/2,d}$ such that*

$$\|\mathfrak{u}(\mathfrak{f}) - \mathfrak{v}(\mathfrak{f})\|_{\sigma/2,d} \leq c|\varepsilon|. \quad (6.14)$$

The mapping $G \ni \mathfrak{f} \to \mathfrak{u}(\mathfrak{f}) \in \mathcal{E}_{\sigma/2,d}$ has a continuous extension to the strip

$$\Sigma_\rho = \{(\alpha, k) \in \mathbb{C}^2 : \ \operatorname{Re} \alpha \in \mathbb{R}, \ \operatorname{Re} k \in [0, 1], \ |\operatorname{Im} \alpha| \leq \varrho, \ |\operatorname{Im} k| \leq \varrho\}.$$

Moreover, the extended mappings

$$\mathfrak{f} \to \boldsymbol{\varphi} - \boldsymbol{\varphi}_0(\alpha), \quad \mathfrak{f} \to M, \quad \mathfrak{f} \to m + \alpha M$$

are 2π- periodic in α.

Proof. In view of Proposition 4.6 for every $\mathfrak{f} \in G$, there is $r > 0$ independent of $\mathfrak{f}$ with the following property. If

$$\|\mathfrak{v}(\mathfrak{f}) - \mathfrak{u}\|_{\sigma,d} \leq r, \quad (6.15)$$

then the operator $\Phi(\mathfrak{f}, \mathfrak{u}) \in \mathcal{F}_{\sigma,0}$ is differentiable at the point $(\mathfrak{f}, \mathfrak{u})$ with respect to $\mathfrak{u}$ and satisfies estimate (6.3) with the exponent $\tau = 0$. On the other

hand, it follows from Proposition 5.4, that there exists $\varepsilon_0 > 0$ independent of $\mathfrak{f}$ with the following properties. If $\mathfrak{f} \in G$, $\mathfrak{u} \in \mathcal{F}_{\sigma,0}$ satisfy inequality (6.15), and $|\varepsilon| \leq \varepsilon_0$, then there is the linear operator $\mathcal{R}(\mathfrak{f}, \mathfrak{u})$ satisfying inequalities (6.5)-(6.7) with the exponents $\tau = 0$ and $\gamma = 8n + 14 + 2d$. In other words, if $(\mathfrak{f}, \mathfrak{u})$ satisfies inequality (6.15) and $|\varepsilon| \leq \varepsilon_0$, then the operator Φ satisfies inequalities (6.3) and (6.5)-(6.7) in Conditions (**H.3**)-(**H.5**) of Theorem 6.1.

Now choose an arbitrary $\mathfrak{f}_0 = (\alpha_0, k_0) \in G$ and set $\mathfrak{u}_0 = \mathfrak{v}(\mathfrak{f}_0)$. It follows from this and (6.12) that

$$\|\mathfrak{v}(\mathfrak{f}) - \mathfrak{u}_0\|_{\sigma,d} \leq 20|\mathfrak{f} - \mathfrak{f}_0| \quad \text{for all} \quad \mathfrak{f} \in G. \tag{6.16}$$

Recall definition (6.2) of the ball $\mathcal{B}_\sigma(\mathfrak{f}_0, \mathfrak{u}_0)$:

$$\mathcal{B}_\sigma(\mathfrak{f}_0, \mathfrak{u}_0) = \{(\mathfrak{f}, \mathfrak{u}) : \ |\mathfrak{f} - \mathfrak{f}_0| < N, \quad \|\mathfrak{u} - \mathfrak{u}_0\|_{\sigma,d} < R\}.$$

We have

$$\|\mathfrak{v}(\mathfrak{f}) - \mathfrak{u}\|_{\sigma,d} \leq \|\mathfrak{u}_0 - \mathfrak{u}\|_{\sigma,d} + \|\mathfrak{v}(\mathfrak{f}) - \mathfrak{u}_0\|_{\sigma,d}$$
$$\leq R + 20|\mathfrak{f} - \mathfrak{f}_0| \leq R + 20N$$

for $(\mathfrak{f}, \mathfrak{u}) \in \mathcal{B}_\sigma$. Hence for

$$R + 20N \leq r, \quad N < \rho \tag{6.17}$$

the vectors $\mathfrak{f}$ and $\mathfrak{u}$ satisfy inequality (6.15) in the ball $\mathcal{B}_\sigma(\mathfrak{f}_0, \mathfrak{u}_0)$. Moreover, we have $\mathcal{B}_\sigma(\mathfrak{f}_0, \mathfrak{u}_0) \cap \mathbb{C}^2 \subset \Sigma_\rho$. Now fix N and R satisfying (6.17). It follows from this and what was mentioned above that for $|\varepsilon| \leq \varepsilon_0$, the operator Φ satisfies Conditions (**H.3**)-(**H.5**) with the exponents $\tau = 0$, $\gamma = 9n + 12 + d$, and the constant c_0 independent of $\mathfrak{f}_0$. Hence Φ meets all requirements of Theorem 6.1. Applying this theorem we conclude that there is a constant C, depending only on c_0 and γ, with the following properties. If the couple $(\mathfrak{f}, \mathfrak{v}(\mathfrak{f})) \in \mathcal{B}_\sigma(\mathfrak{f}_0, \mathfrak{u}_0)$ satisfies the conditions

$$\|\mathfrak{v}(\mathfrak{f}) - \mathfrak{u}_0\|_{\sigma,d} \leq R/2, \quad \|\Phi(\mathfrak{f}, \mathfrak{v}(\mathfrak{f}))\|_{\sigma,0} \leq 2^{-1} C R \sigma^{2\gamma}, \tag{6.18}$$

then equation (6.1) have a unique solution $\mathfrak{u} = \mathfrak{u}(\mathfrak{f}) \in \mathcal{E}_{\sigma/2,d}$ such that

$$\|\mathfrak{v}(\mathfrak{f}) - \mathfrak{u}(\mathfrak{f})\|_{\sigma/2,d} \leq C^{-1} \|\Phi(\mathfrak{f}, \mathfrak{v}(\mathfrak{f}))\|_{\sigma,0} \sigma^{-\gamma}. \tag{6.19}$$

Moreover, the mapping $\mathcal{B}_\sigma(\mathfrak{f}_0, \mathfrak{u}_0) \cap \mathbb{C}^2 \ni \mathfrak{f} \to \mathfrak{u}(\mathfrak{f}) \in \mathcal{E}_{\sigma/2,d}$ is continuous. In order to prove the local existence and uniqueness of solution to equation

(4.16), we have to show that conditions (6.18)is fulfilled in the ball $\mathcal{B}_\sigma(\mathfrak{f}_0, \mathfrak{u}_0)$. First we notice that, in view of relations (6.12) and (4.9) , we have

$$\Phi_1(\mathfrak{f}, \mathfrak{v}(\mathfrak{f})) = \varepsilon H_1(\mathrm{id}, 0, \alpha \mathbf{e}_1),$$

$$\Phi_2(\mathfrak{f}, \mathfrak{v}(\mathfrak{f})) = \varepsilon \frac{\partial H_1}{\partial \mathbf{z}}(\mathrm{id}, 0, \alpha \mathbf{e}_1)\}^\top,$$

$$\Phi_3(\mathfrak{f}, \mathfrak{v}(\mathfrak{f})) = \varepsilon \{\frac{\partial H_1}{\partial \mathbf{y}}(\mathrm{id}, 0, \alpha \mathbf{e}_1)\}^\top,$$

$$\Phi_4(\mathfrak{f}, \mathfrak{v}(\mathfrak{f})) = \varepsilon \frac{\partial^2 H_1}{\partial \mathbf{z}^2}(\mathrm{id}, 0, \alpha \mathbf{e}_1),$$

$$\Phi_5(\mathfrak{f}, \mathfrak{v}(\mathfrak{f})) = 0.$$

It follows from this and analyticity conditions (2.3) imposed on H_1 that

$$\|\Phi(\mathfrak{f}, \mathfrak{v}(f))\|_{\sigma,0} \leq c_1|\varepsilon|$$

for all $\mathfrak{f} \in \Sigma_\rho$. Here the constant c_1 is independent on $\mathfrak{f}$. From this and (6.16) we obtain that condition (6.18) is fulfilled for all $\mathfrak{f}$ satisfying the inequality

$$|\mathfrak{f} - \mathfrak{f}_0| \leq N_1 = N/40, \tag{6.20}$$

and all ε such that

$$\varepsilon \leq \varepsilon_0 < c_1^{-1} C N \sigma^{2\gamma}.$$

Applying Theorem 6.1 we conclude that for all $\mathfrak{f} \in G$ satisfying (6.20) and for all $|\varepsilon| \leq \varepsilon_0$ operator equation (4.16) has a solution $\mathfrak{u}(\mathfrak{f})$ satisfying the inequality

$$\|\mathfrak{v}(\mathfrak{f}) - \mathfrak{u}(\mathfrak{f})\|_{\sigma/2,d} \leq c_1 C^{-1} \sigma^{-\gamma}|\varepsilon| \equiv c_2|\varepsilon|. \tag{6.21}$$

This solution is unique. Moreover, the mapping $\mathbb{C}^2 \ni \mathfrak{f} \to \mathfrak{u}(\mathfrak{f}) \in \mathcal{E}_{\sigma/2,d}$ is continuous in the disk $\{|\mathfrak{f} - \mathfrak{f}_0| \leq N_1\}$.

Our next task is to prove that this solution is defined for all $\mathfrak{f} \in G$. Since G is a compact set, there is a finite collection of the balls

$$G_i = \{|\mathfrak{f} - \mathfrak{f}_i| \leq N_1/2\}, \quad 1 \leq i \leq m, \quad \mathfrak{f}_i \in G$$

such that $G \subset \cup_i G_i$. It follows from the local solvability of operator equation (4.16) that for $|\varepsilon| \leq \varepsilon_0$ there is a unique continuous mapping $G_i \ni \mathfrak{f} \to \mathfrak{u}_i(\mathfrak{f}) \in \mathcal{E}_{\sigma/2,d}$ such that

$$\Phi(\mathfrak{f}, \mathfrak{u}_i(\mathfrak{f})) = 0, \quad \|\mathfrak{u}_i(\mathfrak{f}) - \mathfrak{v}(\mathfrak{f})\|_{\sigma/2,d} \leq c_2|\varepsilon|. \tag{6.22}$$

If $\mathfrak{f} \in G_i \cap G_j$ then $\mathfrak{u}_i(\mathfrak{f}) = \mathfrak{u}_j(\mathfrak{f})$. Indeed, $\mathfrak{u}_j$ satisfies inequality (6.22) and the equation $\Phi(\mathfrak{f}, \mathfrak{u}_j) = 0$. On the other hand, $\mathfrak{u}_i(\mathfrak{f})$ is the unique solution to the equation $\Phi(\mathfrak{f}, \mathfrak{u}) = 0$ with $\mathfrak{f} \in G_i$ satisfying this inequality. Hence $\mathfrak{u}_i = \mathfrak{u}_j$. Therefore, the relation $\mathfrak{u}(\mathfrak{f}) = \mathfrak{u}_i(\mathfrak{f})$ for $\mathfrak{f} \in G_i$ defines the continuous mapping $G \ni \mathfrak{f} \to \mathfrak{u}(\mathfrak{f}) \in \mathcal{E}_{\sigma/2,d}$.

It remains to prove that this mapping has the analytic extension to the strip Σ_ρ. Fix an arbitrary $\mathfrak{f} = (\alpha, k) \in G$ such that

$$\mathrm{Re}\, \alpha \in [0, \pi/2] \tag{6.23}$$

and set

$$\tilde{\alpha} = \alpha + 2\pi, \quad \tilde{\mathfrak{f}} = (\tilde{\alpha}, k).$$

Let $\tilde{\mathfrak{u}} = \mathfrak{u}(\tilde{\mathfrak{f}})$ be a solution to the operator equation $\Phi(\tilde{\mathfrak{f}}, \tilde{\mathfrak{u}}) = 0$. We have

$$\tilde{\mathfrak{u}} = (\tilde{\boldsymbol{\varphi}}, \tilde{e}, \tilde{m}, \tilde{M}), \quad \text{where} \quad \tilde{\boldsymbol{\varphi}} = (\tilde{\boldsymbol{\beta}}, \tilde{\varphi}_0, \tilde{\mathbf{u}}, \tilde{\mathbf{w}}, \tilde{W}_{11}, \tilde{W}_{12}, \tilde{W}_{22}).$$

Now set

$$\mathfrak{u}^* = (\boldsymbol{\varphi}^*, e^*, m^*, M^*),$$

where

$$\boldsymbol{\varphi}^* = \tilde{\boldsymbol{\varphi}} - (0, 0, 0, 2\pi \mathbf{e}_1, 0, 0, 0),$$

$$e^* = \tilde{e} - (2\pi \tilde{m} + 2\pi^2 \tilde{M}), \quad m^* = \tilde{m} + 2\pi \tilde{M}, \quad M^* = \tilde{M}. \tag{6.24}$$

Since the basic Hamiltonian $H(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is 2π-periodic in z_1 it follows from definition (4.9) of the operator Φ that

$$\Phi(\mathfrak{f}, \mathfrak{u}^*) \equiv \Phi(\tilde{\mathfrak{f}}, \tilde{\mathfrak{u}}). \tag{6.25}$$

Next, we have

$$\mathfrak{v}(\tilde{\mathfrak{f}}) = \mathfrak{v}(\mathfrak{f}) - (2\pi k + 2\pi^2 k, -2\pi k, k).$$

Relations (6.24) imply

$$\mathfrak{u}^* - \mathfrak{v}(\mathfrak{f}) = \tilde{\mathfrak{u}} - \mathfrak{v}(\tilde{\mathfrak{f}}) \tag{6.26}$$

Recall that $\tilde{\mathfrak{u}} = \mathfrak{u}(\tilde{\mathfrak{f}})$ satisfies the inequality

$$\|\tilde{\mathfrak{u}} - \mathfrak{v}(\tilde{\mathfrak{f}})\|_{\sigma/2,d} \leq c_2 |\varepsilon|.$$

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and $\Phi(\tilde{\mathfrak{f}}, \tilde{\mathfrak{u}}) = 0$. From this, (6.25) and (6.26) we obtain

$$\Phi(\mathfrak{f}, \mathfrak{u}^*) = 0, \quad \|\mathfrak{u}^* - \mathfrak{v}(\mathfrak{f})\|_{\sigma/2,d} \leq c_2 |\varepsilon|, \quad \mathfrak{f} \in G.$$

Hence $\mathfrak{u}^* = \mathfrak{u}(\mathfrak{f})$, which along with (6.24) and the equality $\tilde{\mathfrak{u}} = \mathfrak{u}(\tilde{\mathfrak{f}})$ yields

$$\boldsymbol{\varphi}(\mathfrak{f}) + 2\pi(0, 0, 0, \mathbf{e}_1, 0, 0, 0) = \boldsymbol{\varphi}(\tilde{\mathfrak{f}}),$$
$$m(\mathfrak{f}) = m(\tilde{\mathfrak{f}}) + 2\pi M(\tilde{\mathfrak{f}}), \quad M(\mathfrak{f}) = M(\tilde{\mathfrak{f}}).$$

We can rewrite these relations in the equivalent form

$$\boldsymbol{\varphi}(\mathfrak{f}) - \alpha(0, 0, 0, \mathbf{e}_1, 0, 0, 0) = \boldsymbol{\varphi}(\tilde{\mathfrak{f}}) - (\alpha + 2\pi)(0, 0, 0, \mathbf{e}_1, 0, 0, 0),$$
$$m(\mathfrak{f}) + \alpha M(\mathfrak{f}) = m(\tilde{\mathfrak{f}}) + (\alpha + 2\pi)M(\tilde{\mathfrak{f}}), \quad M(\mathfrak{f}) = M(\tilde{\mathfrak{f}}). \tag{6.27}$$

Recall that these relations holds true for all $\mathfrak{f}$ and $\tilde{\mathfrak{f}}$ satisfying the conditions

$$\mathfrak{f} = (\alpha, k) \in G, \quad \tilde{\mathfrak{f}} = (\alpha + 2\pi, k), \quad \operatorname{Re} \alpha \in [0, \pi/2].$$

Since the vector-function $\mathfrak{u}(\mathfrak{f})$ and all its components are holomorphic in α, relations (6.27) imply that the mappings

$$\mathfrak{f} \to \boldsymbol{\varphi} - \boldsymbol{\varphi}_0(\alpha), \quad \mathfrak{f} \to M, \quad \mathfrak{f} \to m + \alpha M$$

are holomorphic in the strip Σ_ρ and 2π periodic in α. This completes the proof of Theorem 6.2. $\qquad\square$

7 Dependence on parameters. Jacobi vector fields

7.1 Differentiability with respect to parameters

Theorem 6.2 guarantees the local existence of solutions to the operator equation

$$\Phi(\mathfrak{f}, \mathfrak{u}) = 0. \tag{7.1}$$

In this section we investigate in details the dependence of this solution on the parameter $\mathfrak{f}$. The result is given by the following

Theorem 7.1. *Let conditions* (**H.**1)-(**H.**2) *be satisfied and $\varepsilon_0 > 0$ be given by Theorem 6.2. Let $|\varepsilon| \le \varepsilon_0$ and $\mathfrak{u} = \mathfrak{u}(\mathfrak{f})$ be a solution to equation* (7.1),

$$\mathfrak{u}(\mathfrak{f}) = (\boldsymbol{\varphi}(\alpha, k), e(\alpha, k), m(\alpha, k), M(\alpha, k),$$
$$\boldsymbol{\varphi}(\alpha, k) = (\boldsymbol{\beta}, \varphi_0, \mathbf{u}, \mathbf{w}, W_{11}, W_{12}, W_{21}). \tag{7.2}$$

Then the mappings

$$\mathbb{R} \times [0, 1] \ni (\alpha, k) \to \mathbf{u}(\alpha, k) \in \mathcal{A}_{\sigma/4, d},$$
$$\mathbb{R} \times [0, 1] \ni (\alpha, k) \to \varphi_0(\alpha, k) \in X_{\sigma/4, d},$$
$$\mathbb{R} \times [0, 1] \ni (\alpha, k) \to \mathbf{w}(\alpha, k) - \alpha\mathbf{e}_1 \in \mathcal{A}_{\sigma/2, d},$$
$$\mathbb{R} \times [0, 1] \ni (\alpha, k) \to \mathbf{W}(\alpha, k) \in \mathcal{A}_{\sigma/2, d}, \tag{7.3}$$
$$\mathbb{R} \times [0, 1] \ni (\alpha, k) \to \boldsymbol{\beta}(\alpha, k) \in \mathbb{R}^{n-1},$$
$$\mathbb{R} \times [0, 1] \ni (\alpha, k) \to M(\alpha, k) \in \mathbb{R}^1,$$
$$\mathbb{R} \times [0, 1] \ni (\alpha, k) \to m(\alpha, k) + \alpha M(\alpha, k) \in \mathbb{R}^1,$$

are continuously differentiable and 2π-periodic in α. Moreover, they are analytic in $\mathbb{R} \times (0, 1)$ and satisfy the inequalities

$$\|\partial_\alpha^r \mathbf{u}\|_{\sigma/2, d} + \|\partial_\alpha^r \varphi_0\|_{\sigma/2, d} \le c(r)|\varepsilon|,$$
$$\|\partial_\alpha^r(\mathbf{w} - \alpha\mathbf{e}_1)\|_{\sigma/2, d} + \|\partial_\alpha^r(\mathbf{W} - \mathbf{I})\|_{\sigma/2, d} \le c(r)|\varepsilon|, \tag{7.4}$$
$$|\partial_\alpha^r \boldsymbol{\beta}| + |\partial_\alpha^r(m + \alpha M)| + |\partial_\alpha^r(M + k)| \le c(r)|\varepsilon|,$$

and

$$\|\partial_k \mathbf{u}\|_{\sigma/2, d} + \|\partial_k \varphi_0\|_{\sigma/2, d} \le c|\varepsilon|,$$
$$\|\partial_k(\mathbf{w} - \alpha\mathbf{e}_1)\|_{\sigma/2, d} + \|\partial_k(\mathbf{W} - \mathbf{I})\|_{\sigma/2, d} \le c|\varepsilon|, \tag{7.5}$$
$$|\partial_k \boldsymbol{\beta}| + |\partial_k(m + \alpha M)| + |\partial_k(M + k)| \le c|\varepsilon|,$$

where $r \ge 0$ is an arbitrary integer, the constant c is independent of α, k, and ε.

Proof. By virtue of Theorem (6.2), the mapping $\Sigma_\rho \ni \mathfrak{f} \to \mathfrak{u}(\mathfrak{f}) \in \mathcal{E}_{\sigma/2, 0}$ is continuous in the complex strip Σ_ρ which contains the real axis. Hence this mapping is holomorphic function of α. It is also a holomorphic function of k on the interval $(0, 1)$. Moreover, Theorem 6.2 shows that $\boldsymbol{\beta}, \varphi_0, \mathbf{u}, \mathbf{w} - \alpha\mathbf{e}_1$, $W_{ij}, m + \alpha M$ and M are periodic in α. Since these functions are holomorphic in α on the real axis, estimates (7.4) obviously follows from estimates (6.14) in Theorem 6.2.

However, this theorem does not guarantee the differentiability of $\mathfrak{u}(\alpha, k)$ with respect to k on the closed segment $k \in [0, 1]$. In order to prove estimates (7.5) fix an arbitrary $\alpha \in \mathbb{R}^1$ and consider the function $\mathfrak{u}(\alpha, k)$. In view of Theorem 6.2 this function is continuous in the rectangular $\mathrm{Re}\, k \in [0, 1]$, $|\mathrm{Im}\, k| \leq \rho$. Hence it is holomorphic on the interval $(0, 1)$ and continuous on $[0, 1]$. Next the function $\mathfrak{u}(\alpha, k)$ satisfies the operator equation

$$\Phi(\mathfrak{f}, \mathfrak{u}) \equiv (\Phi_1(\mathfrak{u}), \Phi_2(\mathfrak{u}), \Phi_3(\mathfrak{u}), \Phi_4(k, \mathfrak{u}), \Phi_5(\mathfrak{u})) = 0, \tag{7.6}$$

where the differential operators Φ_i are defined by (4.9). It follows from this relation that the only Φ_4 depends on k via the matrix $\mathbf{\Omega} = \mathrm{diag}\,(-k, 1)$ in the left hand side of (4.9d). Since $\mathfrak{u}$ is analytic function of k on the interval $(0, 1)$, we can differentiate (7.6) with respect to k to obtain

$$D_{\boldsymbol{\varphi}}\Phi(\mathfrak{f}, \mathfrak{u})\partial_k\boldsymbol{\varphi} + D_e\Phi(\mathfrak{f}, \mathfrak{u})\partial_k e +$$
$$D_m\Phi(\mathfrak{f}, \mathfrak{u})\partial_k m + D_M\Phi(\mathfrak{f}, \mathfrak{u})(\partial_k M + 1) = \mathcal{Z} \quad \text{for} \quad k \in (0, 1). \tag{7.7}$$

Here

$$\mathcal{Z} = (0, 0, 0, \mathcal{Z}_4, 0), \quad \mathcal{Z}_4 = \mathrm{diag}\,(1, 0) - \mathbf{W}^\top \mathrm{diag}\,(1, 0)\mathbf{W}.$$

It follows from estimate (7.4) that

$$\|\mathcal{Z}\|_{\sigma/2, 0} \leq c|\varepsilon|. \tag{7.8}$$

Relation (7.7) can be considered as equation for

$$\partial_k \mathfrak{u} = (\partial_k\boldsymbol{\varphi}, \partial_k e, \partial_k m, \partial_k M).$$

Applying to (7.7) Proposition 5.4 with σ, σ_1, and σ' replaced by $\sigma/2$, $\sigma/2$, and $\sigma/4$ we conclude that for a suitable choice of ε_0, equation (7.7) has a unique solution

$$(\partial_k\boldsymbol{\varphi}, \partial_k e, \partial_k m, \partial_k M + 1) = \mathcal{R}(\mathfrak{f}, \mathfrak{u})\mathcal{Z}$$

Estimate (5.13)for the resolvent $\mathcal{R}$ in Proposition (5.4) implies the inequality

$$\|\partial_k\varphi\|_{\sigma/4, d} + |\partial_k m| + |\partial_k M + 1| \leq c\sigma^{-9n-12-d}\|\mathcal{Z}\|_{\sigma/2, 0} \leq c|\varepsilon|,$$

which obviously yields (7.5). This completes the proof of Theorem 7.1. $\quad\square$

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7.2 Representation of derivatives. Jacobi vector fields

In this section we obtain the representation for the derivatives of solutions to the modified problem with respect to the parameters α and k. Let $\boldsymbol{\varphi}(\alpha, k) = (\boldsymbol{\beta}, \varphi_0, \mathbf{u}, \mathbf{w}, W_{11}, W_{12}, W_{21})$ be a solution to the operator equation (7.1) given by Theorem 7.1. Let $\mathbf{V}$, $\mathbf{W}$, and $\mathbf{v}$ are given by (3.4)

$$\mathbf{V} = (\mathbf{I}_{n-1} + \mathbf{u}')^{-\top}, \quad \mathbf{v} = \beta + \mathbf{V}\left(\nabla\varphi_0 - w_2\nabla w_1\right),$$
$$W_{22} = W_{11}^{-1}(1 + W_{12}W_{21}). \tag{7.9}$$

For $\tau = \alpha, k$ denote by $\boldsymbol{\chi}^{(\tau)}$, $\boldsymbol{\lambda}^{(\tau)}$, and $\boldsymbol{\mu}^{(\tau)}$ the functions

$$\boldsymbol{\chi}^{(\tau)} = \mathbf{V}^{\top}\partial_\tau\mathbf{u} \tag{7.10a}$$

$$\boldsymbol{\lambda}^{(\tau)} = \mathbf{W}^{-1}\partial_\tau\mathbf{w} - \chi_i^{(\tau)}\mathbf{W}^{-1}\frac{\partial}{\partial\xi_i}\mathbf{w} \tag{7.10b}$$

$$\boldsymbol{\mu}^{(\tau)} = \mathbf{V}^{-1}\left(\partial_\tau\mathbf{v} + \chi_i^{(\tau)}\frac{\partial}{\partial\xi_i}\mathbf{v} - \boldsymbol{\Lambda}\boldsymbol{\lambda}^{(\tau)}\right). \tag{7.10c}$$

The vector fields $\boldsymbol{\chi}^{(\tau)}$, $\boldsymbol{\lambda}^{(\tau)}$, and $\boldsymbol{\mu}^{(\tau)}$ can be regarded as the Jacobi vector fields for the invariant tori problem. We also set

$$p^{(\tau)} = \partial_\tau m + \alpha\partial_\tau M. \tag{7.11}$$

Here the matrix $\boldsymbol{\Lambda}$ are given by (3.8):

$$\boldsymbol{\Lambda} = -\mathbf{V}\left(\mathbf{w}'\right)\mathbf{J}\,\mathbf{W}.$$

Throughout of this an the next sections we will use the following notation. For every integrable periodic function $f(\xi)$ we set

$$\overline{f} = (2\pi)^{1-n}\int_{\mathbb{T}^{n-1}} f(\xi)\,d\xi, \quad f^* = f - \overline{f}. \tag{7.12}$$

We have the following

Lemma 7.2. *Under the assumptions of Theorem 7.1, the functions* $\boldsymbol{\chi}^{(\tau)}$, $\boldsymbol{\lambda}^{(\tau)}$, $\boldsymbol{\mu}^{(\tau)}$, *and the parameter* $q^{(\tau)}$ *satisfy the equations*

$$\partial\,\boldsymbol{\mu}^{(\tau)} = -p^{(\tau)}\nabla w_1 - \frac{1}{2}\partial_\tau M\,\nabla(w_1^*)^2, \tag{7.13a}$$

$$\partial\,\boldsymbol{\lambda}^{(\tau)} + \boldsymbol{\Omega}\,\boldsymbol{\lambda}^{(\tau)} + \mathbf{T}\,\boldsymbol{\mu}^{(\tau)} + p^{(\tau)}\mathbf{W}^{\top}\boldsymbol{e}_1 = -\,\partial_\tau M w_1^*\mathbf{W}^{\top}\boldsymbol{e}_1, \tag{7.13b}$$

$$-\partial\,\boldsymbol{\lambda}^{(\tau)} + \mathbf{S}\boldsymbol{\mu}^{(\tau)} + \mathbf{T}^{\top}\boldsymbol{\lambda}^{(\tau)} = 0 \tag{7.13c}$$

and the orthogonality conditions

$$\overline{\{W\boldsymbol{\lambda}^{(\tau)} \cdot \mathbf{e}_1\}} + \overline{\{\boldsymbol{\chi}^{(\tau)} \cdot \nabla w_1\}} = \delta_{\alpha\tau}, \tag{7.13d}$$

$$\overline{\{\mathbf{V}^{-\top}\boldsymbol{\chi}^{(\tau)}\}} = 0 \tag{7.13e}$$

Here the matrices $\mathbf{S}$ *and* $\mathbf{T}$ *are defined by* (4.25)

$$\mathbf{S} = \mathbf{V}^\top \frac{\partial^2 H}{\partial \mathbf{y}^2}(\boldsymbol{\xi} + \mathbf{u}, \mathbf{v}, \mathbf{w})\mathbf{V},$$

$$\mathbf{T} = \mathbf{W}^\top \frac{\partial^2 H}{\partial \mathbf{z}\partial \mathbf{y}}(\boldsymbol{\xi} + \mathbf{u}, \mathbf{v}, \mathbf{w})\mathbf{V} + \mathbf{\Lambda}^\top \frac{\partial^2 H}{\partial \mathbf{y}^2}(\boldsymbol{\xi} + \mathbf{u}, \mathbf{v}, \mathbf{w})\mathbf{V}. \tag{7.14}$$

In view of (4.26) *and* (7.3), *they admit the estimates*

$$\|\mathbf{S} - \mathbf{S}_0\|_{\sigma/2, d-1} + \|\mathbf{T} - \mathbf{T}_0\|_{\sigma/2, d-1} \le c|\varepsilon|, \tag{7.15}$$

where the constant matrices $\mathbf{S}_0$, $\mathbf{T}_0$ *are given by*

$$\mathbf{S}_0 = \frac{\partial^2 H_0}{\partial \mathbf{y}^2}(0,0), \quad \mathbf{T}_0 = \frac{\partial^2 H_0}{\partial \mathbf{z}\partial \mathbf{y}}(0,0) \tag{7.16}$$

Proof. Since $\mathfrak{u} = (\boldsymbol{\varphi}, e, m, M)$ is continuously differentiable with respect to $\tau = \alpha, k$ we can differentiate (7.6) with respect to τ to obtain

$$D_{\boldsymbol{\varphi}}\Phi_i(\mathfrak{f}, \mathfrak{u})\partial_\tau\boldsymbol{\varphi} + D_e\Phi_i(\mathfrak{f}, \mathfrak{u})\partial_\tau e +$$
$$D_m\Phi_i(\mathfrak{f}, \mathfrak{u})\partial_\tau m + D_M\Phi(\mathfrak{f}, \mathfrak{u})(\partial_\tau M) + \partial_\tau\Phi_i = 0. \tag{7.17}$$

$i = 1, \ldots, 5$. Introduce the functions

$$\boldsymbol{\chi}^{(\tau)} = \mathbf{V}^\top \partial_\tau \mathbf{u} \tag{7.18a}$$

$$\boldsymbol{\lambda}^{(\tau)} = \mathbf{W}^{-1}\partial_\tau \mathbf{w} - \chi_i^{(\tau)} \mathbf{W}^{-1}\frac{\partial}{\partial \xi_i}\mathbf{w} \tag{7.18b}$$

$$\mathbf{\Gamma}^{(\tau)} = \mathbf{W}^{-1}\partial_\tau \mathbf{W} - \chi_i^{(\tau)} \mathbf{W}^{-1}\frac{\partial}{\partial \xi_i}\mathbf{W}, \tag{7.18c}$$

$$\boldsymbol{\mu}^{(\tau)} = \mathbf{V}^{-1}\left(\partial_\tau \mathbf{v} + \chi_i^{(\tau)}\frac{\partial}{\partial \xi_i}\mathbf{v} - \mathbf{\Lambda}\boldsymbol{\lambda}^{(\tau)}\right) \tag{7.18d}$$

$$\nabla\psi_0^{(\tau)} = \boldsymbol{\mu}^{(\tau)} - \boldsymbol{\beta}^{(\tau)}, \quad \boldsymbol{\beta}^{(\tau)} = \frac{1}{(2\pi)^{n-1}}\int_{\mathbb{T}^{n-1}}\boldsymbol{\mu}^{(\tau)}\, d\boldsymbol{\xi}, \tag{7.18e}$$

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Since $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{W}$ form a canonical mapping in the group $\mathcal{G}$, the vector functions $\partial_\tau \mathbf{u}$, $\partial_\tau \mathbf{v}$, $\partial_\tau \mathbf{w}$, and $\partial_\tau \mathbf{W}$ determine the element of the tangent space of $\mathcal{G}$. Hence we can apply Corollary 4.8 of the structural theorem 4.7 to obtain

$$D_\varphi \Phi_1[\partial_\tau \boldsymbol{\varphi}] + D_e \Phi_1[\partial_\tau e] + D_m \Phi_1[\partial_\tau m] + D_M \Phi_1[\partial_\tau M] = \qquad (7.19a)$$

$$\boldsymbol{\partial}\psi_0^{(\tau)} + \boldsymbol{\omega}^\top \cdot \boldsymbol{\beta}^\tau + \partial_\tau e + \partial_\tau m\, w_1 + \partial_\tau M\, \frac{1}{2}w_1^2,$$

$$D_\varphi \Phi_2[\partial_\tau \boldsymbol{\varphi}] + D_e \Phi_2[\partial_\tau e] + D_m \Phi_2[\partial_\tau m] + D_M \Phi_b[\partial_\tau M] \equiv \qquad (7.19b)$$

$$\mathbf{J}\boldsymbol{\partial}\boldsymbol{\lambda}^{(\tau)} + \boldsymbol{\Omega}\boldsymbol{\lambda}^{(\tau)} + \mathbf{T}\boldsymbol{\mu}^{(\tau)} + \partial_\tau m\mathbf{W}^\top\mathbf{e}_1 + \partial_\tau M w_1 \mathbf{W}^\top\mathbf{e}_1,$$

$$\boldsymbol{\mu}^{(\tau)} = \nabla\psi_0^{(\tau)} + \boldsymbol{\beta}^{(\tau)}$$

$$D_\varphi \Phi_3[\partial_\tau \boldsymbol{\varphi}] + D_e \Phi_3[\partial_\tau e] + D_m \Phi_3[\partial_\tau m] + D_M \Phi_M[\partial_\tau M] \equiv \qquad (7.19c)$$

$$-\boldsymbol{\partial}\boldsymbol{\chi}^{(\tau)} + \mathbf{S}\boldsymbol{\mu}^{(\tau)} + \mathbf{T}^\top\boldsymbol{\lambda}^{(\tau)},$$

$$D_\varphi \Phi_5[\partial_\tau \boldsymbol{\varphi}] + D_e \Phi_5[\partial_\tau e] + D_m \Phi_3[\partial_\tau m] + D_M \Phi_M[\partial_\tau M] \equiv \qquad (7.19d)$$

$$\overline{\boldsymbol{\lambda}^{(\tau)} \cdot \mathbf{W}^\top\mathbf{e}_1} \; + \; \overline{\boldsymbol{\chi}^{(\tau)} \cdot \nabla w_1}$$

Since $D_\tau \Phi_1 = 0$ we have from equality (7.17)

$$\boldsymbol{\partial}\psi_0^{(\tau)} + \boldsymbol{\omega}^\top \cdot \boldsymbol{\beta}^{(\tau)} + \partial_\tau e + \partial_\tau m\, w_1 + \partial_\tau M\, \frac{1}{2}w_1^2 = 0 \qquad (7.20)$$

Next we have

$$\nabla\left(\boldsymbol{\partial}\psi_0^{(\tau)} + \boldsymbol{\omega}^\top \cdot \boldsymbol{\beta}^{(\tau)}\right) = \boldsymbol{\partial}\boldsymbol{\mu}^{(\tau)}. \qquad (7.21)$$

On the other hand, relations $p^{(\tau)} = \partial_\tau m + \alpha \partial_\tau M$ and $w_1 = \alpha + w_1^*$ yield

$$\nabla(\partial_\tau m\, w_1 + \partial_\tau M\, \frac{1}{2}w_1^2) = p^{(\tau)}\nabla w_1^* + \partial_\tau M \frac{1}{2}\nabla(w_1^*)^2 \qquad (7.22)$$

Taking the gradient from both the sides of (7.20) and using equalities (7.21)-(7.22) we arrive at (7.13a). Since $\partial_\tau \Phi_2 = 0$ it follows from (7.17) and (7.19b) that

$$\mathbf{J}\boldsymbol{\partial}\boldsymbol{\lambda}^{(\tau)} + \boldsymbol{\Omega}\boldsymbol{\lambda}^{(\tau)} + \mathbf{T}\boldsymbol{\mu}^{(\tau)} + \partial_\tau m\mathbf{W}^\top\mathbf{e}_1 + \partial_\tau M w_1 \mathbf{W}^\top\mathbf{e}_1 = 0. \qquad (7.23)$$

It is easily seen that

$$\partial_\tau m\mathbf{W}^\top\mathbf{e}_1 + \partial_\tau M w_1\mathbf{W}^\top\mathbf{e}_1 = p^{(\tau)}\mathbf{W}^\top\mathbf{e}_1 + w_1^*\mathbf{W}^\top\mathbf{e}_1.$$

Substituting this relation into (7.23) we arrive at (7.13b). Next (7.17) and (7.19c) yield (7.13c) since $\partial_\tau \Phi_3 = 0$. Next, we have $\partial_\tau \Phi_5 = -\delta_{\tau\alpha}$. Substituting this equality and relation (7.19d) into (7.17) we arrive at (7.13d). It remains to note that equation (7.13e) obviously follows from the equality $\overline{\partial_\tau \mathbf{u}} = 0$ and relation (7.18a).

$\square$

Corollary 7.3. *Under the assumptions of Theorem 7.1, we have*

$$(\boldsymbol{\mu}^{(\alpha)}, \boldsymbol{\lambda}^{(\alpha)}, \boldsymbol{\chi}^{(\alpha)}, p^{(\alpha)}) = (\boldsymbol{\mu}^{(1)}, \boldsymbol{\lambda}^{(1)}, \boldsymbol{\chi}^{(1)}, p^{(1)}) + \partial_\alpha M(\boldsymbol{\mu}^{(2)}, \boldsymbol{\lambda}^{(2)}, \boldsymbol{\chi}^{(2)}, p^{(2)})$$
$$(\boldsymbol{\mu}^{(k)}, \boldsymbol{\lambda}^{(k)}, \boldsymbol{\chi}^{(k)}, p^{(k)}) = \partial_k M(\boldsymbol{\mu}^{(2)}, \boldsymbol{\lambda}^{(2)}, \boldsymbol{\chi}^{(2)}, p^{(2)}),$$
$$(7.24)$$

Where $(\boldsymbol{\mu}^{(i)}, \boldsymbol{\lambda}^{(i)}, \boldsymbol{\chi}^{(i)}, q^{(i)})$, $i = 1, 2$ are solutions to the equations

$$\boldsymbol{\partial}\,\boldsymbol{\mu}^{(1)} = -p^{(1)}\nabla w_1, \tag{7.25a}$$

$$\mathbf{J}\boldsymbol{\partial}\,\boldsymbol{\lambda}^{(1)} + \boldsymbol{\Omega}\,\boldsymbol{\lambda}^{(1)} + \mathbf{T}\,\boldsymbol{\mu}^{(1)} + p^{(1)}\mathbf{W}^\top \boldsymbol{e}_1 = 0, \tag{7.25b}$$

$$-\boldsymbol{\partial}\,\boldsymbol{\lambda}^{(1)} + \mathbf{S}\boldsymbol{\mu}^{(1)} + \mathbf{T}^\top \boldsymbol{\lambda}^{(1)} = 0 \tag{7.25c}$$

$$\overline{\{W\boldsymbol{\lambda}^{(1)} \cdot \mathbf{e}_1\}} + \overline{\{\boldsymbol{\chi}^{(1)} \cdot \nabla w_1\}} = 1, \tag{7.25d}$$

$$\overline{\{\mathbf{V}^{-\top}\boldsymbol{\chi}^{(1)}\}} = 0, \tag{7.25e}$$

and

$$\boldsymbol{\partial}\,\boldsymbol{\mu}^{(2)} = -p^{(2)}\nabla w_1 - \frac{1}{2}\nabla(w_1^*)^2, \tag{7.26a}$$

$$\mathbf{J}\boldsymbol{\partial}\,\boldsymbol{\lambda}^{(2)} + \boldsymbol{\Omega}\,\boldsymbol{\lambda}^{(2)} + \mathbf{T}\,\boldsymbol{\mu}^{(2)} + p^{(2)}\mathbf{W}^\top \boldsymbol{e}_1 = -w_1^*\mathbf{W}^\top \boldsymbol{e}_1, \tag{7.26b}$$

$$-\boldsymbol{\partial}\,\boldsymbol{\lambda}^{(2)} + \mathbf{S}\boldsymbol{\mu}^{(2)} + \mathbf{T}^\top \boldsymbol{\lambda}^{(2)} = 0 \tag{7.26c}$$

$$\overline{\{W\boldsymbol{\lambda}^{(2)} \cdot \mathbf{e}_1\}} + \overline{\{\boldsymbol{\chi}^{(2)} \cdot \nabla w_1\}} = 0, \tag{7.26d}$$

$$\overline{\{\mathbf{V}^{-\top}\boldsymbol{\chi}^{(2)}\}} = 0 \tag{7.26e}$$

In the next section we investigate problems (7.25) and (7.26) in many details.

8 Linear problem

In this section we prove the solvability of problems (7.25) and (7.26) and establish estimates for their solutions in the spaces of analytic functions and the Sobolev spaces. First we recall the basic facts from the theory of Sobolev spaces on torus.

8.1 Preliminaries. Sobolev spaces

For every $s \in \mathbb{R}$, we denote by H_s the Hilbert space which consists of all distributions u on the tori $\mathbb{T}^{n-1}$ such that

$$|u|_s = \Big(\sum_{m \in \mathbb{Z}^{n-1}} (1 + |m|^2)^s |\hat{u}(m)|^2 \Big)^{1/2}, \tag{8.1}$$

where $\hat{u}$ is the Fourier transform of u. If u is an integrable function, then

$$\hat{u}(m) = (2\pi)^{-(n-1)/2} \int_{\mathbb{T}^{n-1}} e^{-im \cdot \boldsymbol{\xi}} \, u(\boldsymbol{\xi}) \, d\boldsymbol{\xi}.$$

It is clear that $H_0 = L^2(\mathbb{T}^{n-1})$ and

$$|u|_s = \|(-\Delta + 1)^{s/2} u\|_{L^2(\mathbb{T}^{n-1})}.$$

For every $s' < s''$ the embedding $H_{s''} \hookrightarrow H_{s'}$ is compact. For every integer $l \geq 0$ the embedding $H_{n-1+l} \hookrightarrow C^l(\mathbb{T}^{n-1})$ is compact. Next, we have

$$|uv|_s \leq c(s)|u|_s \|v\|_s \quad \text{for} \quad s \geq n - 1. \tag{8.2}$$

For all $-\infty < r \leq s < \infty$ we have the interpolation inequality

$$\|u\|_{(s+r)/2} \leq c(s, r)\|u\|_s^{1/2}\|u\|_r^{1/2}. \tag{8.3}$$

Recall the estimate $|\hat{u}(m)| \leq \|u\|_{\sigma,0} \exp(-\sigma|m|)$. It follows that

$$|u|_s \leq c(s, \sigma)\|u\|_{\sigma,0} \quad \text{for all} \quad s \in \mathbb{R} \quad \text{and} \quad \sigma > 0. \tag{8.4}$$

8.2 Basic linear problem.

In this section we prove the solvability of linear problems (7.25) and (7.26). We investigate the qualitative properties of their solutions. First we consider the general boundary value problem which includes problems (7.25) and (7.26) as particular cases.

$$\boldsymbol{\partial}\,\boldsymbol{\mu} \;=\; -p\nabla w_1 - \nabla g_1, \tag{8.5a}$$

$$\mathbf{J}\boldsymbol{\partial}\,\boldsymbol{\lambda} \;+\; \boldsymbol{\Omega}\,\boldsymbol{\lambda} \;+\; \mathbf{T}\,\boldsymbol{\mu} + p\mathbf{W}^{\top}\mathbf{e}_1 = \mathbf{g}_2 \tag{8.5b}$$

$$-\boldsymbol{\partial}\,\boldsymbol{\lambda} \;+\; \mathbf{S}\boldsymbol{\mu} \;+\; \mathbf{T}^{\top}\boldsymbol{\lambda} = 0 \tag{8.5c}$$

$$\overline{\{W\boldsymbol{\lambda}\}\cdot \mathbf{e}_1} \;+\; \overline{\{\boldsymbol{\chi}\cdot\nabla w_1\}} = \gamma, \tag{8.5d}$$

$$\overline{\{\mathbf{V}^{-\top}\boldsymbol{\chi}\}} \;=\; 0. \tag{8.5e}$$

Assume that w_1 and $\mathbf{W}$ meet all requirements of Theorem 7.1, i.e.

$$\|\mathbf{V}^{-1} - \mathbf{I}\|_{\sigma/2,0} + \|(w_1 - \alpha)\|_{\sigma/2,d} + \|\mathbf{W} - \mathbf{I}\|_{\sigma/2,d} \le c|\varepsilon|. \tag{8.6}$$

We also assume that the matrix -valued functions $\mathbf{S}$ and $\mathbf{T}$ meet all requirements of Lemma 7.2, i.e.

$$\|\mathbf{S} - \mathbf{S}_0\|_{\sigma/2,d-1} + \|\mathbf{T} - \mathbf{T}_0\|_{\sigma/2,d-1} \le c|\varepsilon|, \tag{8.7}$$

where the constant matrices $\mathbf{S}_0$, $\mathbf{T}_0$ are given by

$$\mathbf{S}_0 = \frac{\partial^2 H_0}{\partial \mathbf{y}^2}(0,0), \quad \mathbf{T}_0 = \frac{\partial^2 H_0}{\partial \mathbf{z}\partial \mathbf{y}}(0,0) \tag{8.8}$$

Recall that $d \ge 2$ is a fixed number. The following theorem is the main result of this section.

Theorem 8.1. *Let $s \in \mathbb{R}^1$ be an arbitrary number and $\sigma \in (1/2, 1]$. Then there is $\varepsilon_0 > 0$ independent on α and k with the following properties. For every $|\varepsilon| \le \varepsilon_0$, $g_1, \mathbf{g}_2 \in \mathcal{A}_{\sigma/2,0}$ and $\gamma \in \mathbb{R}^1$, problem (8.5) has a unique solution $(\boldsymbol{\mu}, \boldsymbol{\lambda}, \boldsymbol{\chi}) \in \mathcal{A}_{\sigma/4,0}$, $p \in \mathbb{R}^1$. This solution admits the estimates*

$$\|\boldsymbol{\mu}\|_{\sigma/4,0} + \|\boldsymbol{\lambda}\|_{\sigma/4,0} + \|\boldsymbol{\chi}\|_{\sigma/4,0} + |p| \le c\big(\|g_1\|_{\sigma/2,0} + \|\mathbf{g}_2\|_{\sigma/2,0} + |\gamma|\big), \tag{8.9}$$

$$|\boldsymbol{\mu}|_s + |\boldsymbol{\lambda}|_s + |\boldsymbol{\chi}|_s + |p| \le c\big(|g_1|_{s+4n+6} + |\mathbf{g}_2|_{s+4n+6} + |\gamma|\big). \tag{8.10}$$

$$|\overline{\boldsymbol{\mu}}| \le c|\overline{\mathbf{g}_2}| + \varepsilon_0 c\big(|g_1|_{s+4n+6} + |\mathbf{g}_2|_{s+4n+6} + |\gamma|\big). \tag{8.11}$$

Proof. The proof is in Appendix E.2. $\qquad\square$

We employ Theorem 8.1 in order to investigate the structure of solutions to problems (7.25) and (7.26) and to establish robust estimates of these solutions. We start with detailed analysis of problem (7.26). The corresponding results is given by the following theorem which is the second main statement of this section.

Theorem 8.2. *Under the assumptions of of Theorem 8.1, there exists $\varepsilon_0 > 0$ with the following property. For every $|\varepsilon| \leq \varepsilon_0$, problem (7.26) has a unique analytic solution $(\boldsymbol{\mu}^{(2)}, \boldsymbol{\lambda}^{(2)}, \boldsymbol{\chi}^{(2)}) \in \mathcal{A}_{\sigma/4,0}$, $p^{(2)} \in \mathbb{C}$. This solution admits the estimate*

$$\|\boldsymbol{\mu}^{(2)}\|_{\sigma/4,0} + \|\boldsymbol{\lambda}^{(2)}\|_{\sigma/4,0} + \|\boldsymbol{\chi}^{(2)}\|_{\sigma/4,0} + |p^{(2)}| \leq c\|w_1^*\|_{\sigma/2,0} \leq c|\varepsilon|, \quad (8.12)$$

$$|\boldsymbol{\mu}^{(2)}|_0 \leq c\varepsilon_0 \, |w_1^*|_{-1}, \quad |\lambda_2^{(2)}|_0 + |k| \, |\lambda_1^{(2)}|_0 \geq c^{-1} \, |w_1^*|_{-1}. \quad (8.13)$$

Here the strictly positive constant c is independent of α, k, and ε.

Proof. Notice that problem (7.26) is a particular case of problem (8.5) with the right hand sides

$$g_1* = (w_1^*)^2, \quad \mathbf{g}_2 = -w_1^*\mathbf{We}_1, \quad \gamma = 0. \quad (8.14)$$

It follows from inequalities (8.6) and the identity $w_1^* = w_1 - \alpha$ that

$$\|g_1^*\|_{\sigma/2,0} \leq c\|w_1^*\|_{\sigma/2,0}^2, \quad \|\mathbf{g}_2\|_{\sigma/2,0} \leq c\|w_1^*\|_{\sigma/2,0}.$$

Applying Theorem 8.1 we conclude that problem (7.26) has a unique solution, satisfying inequality (8.12). It remains to prove estimates (8.13) Choose an arbitrary $t > n$. Since the space H_{t+1} is a Banach algebra, we have

$$|\nabla(w_1^*)^2|_t \leq |(w_1^*)^2|_{t+1} \leq c|w_1^*|_{t+1}^2.$$

On the other hand, the interpolation inequality implies

$$|w_1^*|_{t+1} \leq c|w_1^*|_{-1}^{1/2} \, |w_1^*|_{2t+3}^{1/2} \leq c|w_1^*|_{-1}^{1/2} \, \|w_1^*\|_{\sigma/2,0}^{1/2} \leq c\varepsilon_0^{1/2} \, |w_1^*|_{-1}^{1/2}.$$

We thus get

$$|\nabla(w_1^*)^2|_t \leq c|\varepsilon| \, |w_1^*|_{-1}. \quad (8.15)$$

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Obviously, this inequality holds true for all $t \in \mathbb{R}$. Next we have

$$|w_1^* \mathbf{W} \mathbf{e}_1|_t \leq c\|\mathbf{W}\|_{\sigma/2,0} |w_1^*|_t \leq c|w_1^*|_t. \tag{8.16}$$

Now set $s = -1 - 4n - 6$ and $t = -1$. Apply Theorem 8.1 to problem (7.26). It follows from the estimate (8.10) in this theorem and relations (8.13)-(8.16) that

$$|\boldsymbol{\mu}^{(2)}|_{-4n-7} + |\boldsymbol{\lambda}^{(2)}|_{-4n-7} + |\boldsymbol{\chi}^{(2)}|_{-4n-7} + |p^{(2)}| \leq c|\mathbf{w}_1^*|_{-1}. \tag{8.17}$$

Since $s + 4n + 6 = -1$, estimate (8.11) implies

$$|\overline{\boldsymbol{\mu}^{(2)}}| \leq |\overline{\mathbf{g}_2}| + c\varepsilon_0|\mathbf{w}_1^*|_{-1}.$$

We have

$$|\overline{\mathbf{g}_2}| = |\overline{w_1^* \mathbf{W} \mathbf{e}_1}| = |\overline{w_1^*(\mathbf{W} - \mathbf{I})\mathbf{e}_1}| \leq |w_1^*|_{-1} |\mathbf{W} - \mathbf{I}|_1 \leq c\varepsilon_0|w_1^*|_{-1}.$$

Combining the obtained results we arrive at the estimate

$$|\overline{\boldsymbol{\mu}^{(2)}}| \leq c\varepsilon_0|\mathbf{w}_1^*|_{-1} \tag{8.18}$$

Let us estimate $\boldsymbol{\mu}^{(2)*}$. The first equation (7.26a) in system (7.26) reads

$$\boldsymbol{\partial} \boldsymbol{\mu}^{(2)*} = -p^{(2)}\nabla w_1 - \frac{1}{2}\nabla(w_1^*)^2$$

Applying Lemma E.2 to this equation we obtain

$$|\boldsymbol{\mu}^{(2)*}|_0 \leq c|p^{(2)}| \, |w_1^*|_{4n+7} + c|\nabla(w_1^*)^2|_{4n+6}$$

It follows from inequality (8.15) with $t = 4n + 6$ that

$$|w_1^*|_{4n+7} \leq c\|w_1^*\|_{\sigma/2,0} \leq c\varepsilon_0, \quad |\nabla(w_1^*)^2|_{4n+6} \leq c\varepsilon_0|w_1^*|_{-1}.$$

It follows that

$$|\boldsymbol{\mu}^{(2)*}|_0 \leq c\varepsilon_0|w_1^*|_{-1}.$$

Combining this result with (8.18) we obtain the first estimate in (8.13). Let us prove the second. Notice that $|\boldsymbol{\partial}\lambda_2^{(2)}|_{-1} \leq c|\boldsymbol{\lambda}_2^{(2)}|_0$. It follows that

$$|\lambda_2^{(2)}|_0 + |k| \, |\lambda_1^{(2)}|_0 \geq c^{-1}(|\boldsymbol{\partial}\lambda_2^{(2)}|_{-1} + |k| \, |\lambda_1^{(2)}|_{-1}) \geq$$
$$c^{-1}(|\boldsymbol{\partial}\lambda_2^{(2)} - k \, \lambda_1^{(2)}|_{-1}) \geq c^{-1}(|\boldsymbol{\partial}\lambda_2^{(2)*} - k \, \lambda_1^{(2)*}|_{-1}). \tag{8.19}$$

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On the other hand, the second equation (7.26b) in system (7.26) implies

$$\partial \lambda_2^{(2)^*} - k\,\lambda_1^{(2)^*} = -w_1^* - (\mathbf{T}\boldsymbol{\mu}^{(2)})^* - p^{(2)}(\mathbf{W}^\top - \mathbf{I})^* \mathbf{e}_1 - (w_1^*(\mathbf{W}^\top - \mathbf{I})\mathbf{e}_1)^*.$$

We have

$$|(\mathbf{W}^\top - \mathbf{I})^*|_{-1} \leq \|(\mathbf{W}^\top - \mathbf{I})^*\|_{\sigma/2,0} \leq c\varepsilon_0,$$
$$|w_1^*(\mathbf{W}^\top - \mathbf{I})\mathbf{e}_1|_{-1} \leq c|w_1^*|_{-1}\|(\mathbf{W}^\top - \mathbf{I})^*\|_{\sigma/2,0} \leq c\varepsilon_0|w_1^*|_{-1}.$$

Next, the first estimate in (8.13) implies

$$|\mathbf{T}\boldsymbol{\mu}^{(2)}|_0 \leq c\varepsilon_0\,|w_1^*|_{-1}.$$

Combining obtained inequalities we arrive at the estimate

$$|w_1^* + (\mathbf{T}\boldsymbol{\mu}^{(2)})^* + p^{(2)}(\mathbf{W}^\top - \mathbf{I})^* \mathbf{e}_1 + (w_1^*(\mathbf{W}^\top - \mathbf{I})\mathbf{e}_1)^*|_{-1} \geq |w_1^*|_{-1} - c\varepsilon_0|w_1^*|_{-1},$$

which yields
$$|\partial \lambda_2^{(2)^*} - k\,\lambda_1^{(2)^*}|_{-1} \geq |w_1^*|_{-1} - c\varepsilon_0|w_1^*|_{-1}.$$

This estimate along with (8.19) implies the estimate

$$|\lambda_2^{(2)}|_0 + |k|\,|\lambda_1^{(2)}|_0 \geq c^{-1}(|w_1^*|_{-1} - c\varepsilon_0|w_1^*|_{-1}).$$

Choosing ε_0 sufficiently small we obtain the second estimate in (8.13) and the theorem follows. □

Now turn to problem (7.25). Our goal is to prove the existence and uniqueness of solutions to this problem and to obtain the asymptotic expansion of its solutions near $\varepsilon = 0$. In order to formulate the results we introduce the auxiliary denotations. Recall denotations (7.14)-(7.16) for the matrix $\mathbf{T}$. Denote by $\mathbf{t}_i^\top$, $i = 1, 2$, the rows of $\mathbf{T}$. They can be regarded al analytic periodic functions $\mathbf{t}_i : \mathbb{T}^{n-1} \to \mathbb{C}^{n-1}$. Notice that for $\varepsilon = 0$, we have $\mathbf{t}_1 = 0$ and $\mathbf{t}_2 = \mathbf{t}_0$, where $\mathbf{t}_0$ is the only nonzero row of the matrix $\mathbf{T}_0$. It follows from this and estimate (7.15) that

$$\|\mathbf{t}_1\|_{\sigma/2,d-1} + \|\mathbf{t}_2 - \mathbf{t}_0\|_{\sigma/2,d-1} \leq c|\varepsilon|. \tag{8.20}$$

Recall the denotation

$$\overline{\mathbf{t}_1} = \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{T}^{n-1}} \mathbf{t}_1\, d\boldsymbol{\xi}.$$

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Introduce the constant vector

$$\boldsymbol{\mu}_0 = \mathbf{K}_0^{-1}\overline{\mathbf{t}_1}. \tag{8.21}$$

Our considerations are based on the following algebraic lemma.

Lemma 8.3. *Let $k = 0$. There is $\varepsilon_0 > 0$ with the following properties. For every $|\varepsilon| \leq \varepsilon_0$, the system of linear equations*

$$\begin{aligned}
\boldsymbol{\Omega}\boldsymbol{\lambda}_1 + \overline{\mathbf{T}}\,\boldsymbol{\mu}_1 + p_1\,\mathbf{e}_1 &= 0, \\
\overline{\mathbf{T}}^{\top}\boldsymbol{\lambda}_1 + \overline{\mathbf{S}}\,\boldsymbol{\mu}_1 &= 0, \\
\boldsymbol{\lambda}_1\mathbf{e}_1 &= 1.
\end{aligned} \tag{8.22}$$

has a unique solution $\boldsymbol{\lambda}_1 \in \mathbb{C}^2$, $\boldsymbol{\mu}_1 \in \mathbb{C}^{n-1}$, $p_1 \in \mathbb{C}$ such that

$$\boldsymbol{\mu}_1 = \boldsymbol{\mu}_0 + \boldsymbol{\mu}_2, \quad \boldsymbol{\lambda}_1 = \mathbf{e}_1 - (\mathbf{t}_0 \cdot \boldsymbol{\mu}_0)\mathbf{e}_2 + \boldsymbol{\lambda}_2 \tag{8.23}$$

$$|\boldsymbol{\mu}_2| \leq c|\varepsilon|\,|\boldsymbol{\mu}_0|, \quad |p_1| \leq c|\varepsilon|\,|\boldsymbol{\mu}_0|, \quad |\boldsymbol{\lambda}_2| \leq c|\varepsilon|\,|\boldsymbol{\mu}_0|. \tag{8.24}$$

Proof. Rewrite the first two equations in the form

$$\begin{aligned}
\boldsymbol{\Omega}\boldsymbol{\lambda}_1 + \mathbf{T}_0\,\boldsymbol{\mu}_1 + p_1\,\mathbf{e}_1 &= (\mathbf{T}_0 - \overline{\mathbf{T}})\boldsymbol{\mu}_1, \\
\mathbf{T}_0^{\top}\,\boldsymbol{\lambda}_1 + \mathbf{S}_0\,\boldsymbol{\mu}_1 &= (\mathbf{T}_0^{\top} - \overline{\mathbf{T}}^{\top})\,\boldsymbol{\lambda}_1 + (\mathbf{S}_0 - \overline{\mathbf{S}})\,\boldsymbol{\mu}_1.
\end{aligned}$$

Notice that $\boldsymbol{\lambda}_1 = \lambda_{1,2}\mathbf{e}_2 + \mathbf{e}_1$. Thus we get the linear system of the equations for $\lambda_{1,2}$ and $\boldsymbol{\mu}_1$

$$\begin{aligned}
\lambda_{1,2} + \mathbf{t}_0 \cdot \boldsymbol{\mu}_1 &= \{(\mathbf{T}_0 - \overline{\mathbf{T}})\boldsymbol{\mu}_1\} \cdot \mathbf{e}_2 \\
\mathbf{S}_0\boldsymbol{\mu}_1 + \lambda_{1,2}\,\mathbf{t}_0 &= (\mathbf{T}_0^{\top} - \overline{\mathbf{T}}^{\top})\mathbf{e}_1 + \\
\lambda_{1,2}(\mathbf{T}_0^{\top} - \overline{\mathbf{T}}^{\top})\mathbf{e}_2 &+ (\mathbf{S}_0 - \overline{\mathbf{S}})\boldsymbol{\mu}_1.
\end{aligned} \tag{8.25}$$

Express $\lambda_{1,2}$ in terms of $\boldsymbol{\mu}_1$ using the first equation in (8.25). Substituting the result into the second equation in (8.25) we obtain the following equation for $\boldsymbol{\mu}_1$

$$\begin{aligned}
\mathbf{K}_0\boldsymbol{\mu}_1 = (\mathbf{S}_0 - \overline{\mathbf{S}})\boldsymbol{\mu}_1 + \{(\mathbf{T}_0 - \overline{\mathbf{T}})\boldsymbol{\mu}_1\}_2\big((\mathbf{T}_0^{\top} - \overline{\mathbf{T}}^{\top})\mathbf{e}_2 - \mathbf{t}_0\big) \\
-(\boldsymbol{\mu}_1 \cdot \mathbf{t}_0)(\mathbf{T}_0^{\top} - \overline{\mathbf{T}}^{\top})\mathbf{e}_2 + (\mathbf{T}_0^{\top} - \overline{\mathbf{T}}^{\top})\mathbf{e}_1,
\end{aligned}$$

where the matrix $\mathbf{K}_0 = \mathbf{S}_0 - \mathbf{t}_0 \otimes \mathbf{t}_0$ has a bounded inverse. Using the identities $(\mathbf{T}_0^\top - \overline{\mathbf{T}}^\top)\boldsymbol{e}_2 - \mathbf{t}_0 = -\overline{\mathbf{t}}_2$ and $(\mathbf{T}_0^\top - \overline{\mathbf{T}}^\top)\boldsymbol{e}_1 = \overline{\mathbf{t}}_1$, we can rewrite this equation in the equivalent form

$$\mathbf{K}_0\boldsymbol{\mu}_1 = (\mathbf{S}_0 - \overline{\mathbf{S}})\boldsymbol{\mu}_1 - \{(\mathbf{T}_0 - \overline{\mathbf{T}})\boldsymbol{\mu}_1\}_2\,\overline{\mathbf{t}}_2$$
$$-(\boldsymbol{\mu}_1 \cdot \mathbf{t}_0)(\mathbf{T}_0^\top - \overline{\mathbf{T}}^\top)\boldsymbol{e}_2 - \overline{\mathbf{t}}_1$$

Thus we get the following equation for $\boldsymbol{\mu}_0$

$$\boldsymbol{\mu}_1 - \mathbf{A}\boldsymbol{\mu}_1 = -\boldsymbol{\mu}_0, \tag{8.26}$$

where the linear mapping $\mathbf{A} : \mathbb{C}^2 \to \mathbb{C}^2$ is given by

$$\mathbf{A} : \boldsymbol{\mu}_1 \mapsto \mathbf{K}_0^{-1}\Big\{(\mathbf{S}_0 - \overline{\mathbf{S}})\boldsymbol{\mu}_1 - \{(\mathbf{T}_0 - \overline{\mathbf{T}})\boldsymbol{\mu}\}_2\,\overline{\mathbf{t}}_2 - (\boldsymbol{\mu}_1 \cdot \mathbf{t}_0)\big((\mathbf{T}_0^\top - \overline{\mathbf{T}}^\top)\boldsymbol{e}_2\Big\}.$$

In view of estimates (7.14), the mapping $\mathbf{A}$ admits the estimate

$$|\mathbf{A}\boldsymbol{\mu}_1| \leq c|\varepsilon|\,|\boldsymbol{\mu}_1|. \tag{8.27}$$

Choosing ε_0 sufficiently small we obtain that equation (8.27) has a unique solution which admits decomposition estimate (8.23) with the reminder

$$\boldsymbol{\mu}_2 = (\mathbf{I} - (\mathbf{I} - \mathbf{A})^{-1})\boldsymbol{\mu}_0.$$

When equation (8.27) is solved the vector $\boldsymbol{\lambda}_1$ and the scalar p_1 are restored by the relations

$$\boldsymbol{\lambda}_1 = \mathbf{e}_1 - (\mathbf{t}_0 \cdot \boldsymbol{\mu}_1)\mathbf{e}_2$$
$$p_1 = (\mathbf{T}_0 - \overline{\mathbf{T}})\boldsymbol{\mu}_1 \cdot \mathbf{e}_1 \tag{8.28}$$

which gives decomposition (8.23) with the reminder

$$\boldsymbol{\lambda}_2 = -(\mathbf{t}_0 \cdot \boldsymbol{\mu}_1)\mathbf{e}_2$$

Estimate (8.24) for $\boldsymbol{\lambda}_2$ and p_1 obviously follows from estimate (8.24) for $\boldsymbol{\mu}_2$ and the decomposition (8.23) for $\boldsymbol{\mu}_1$. $\square$

The following theorem, which is the third main result of this section, constitutes the properties of solutions to problem (7.26)

Theorem 8.4. *Under the assumptions of of Theorem 8.1, there exists $\varepsilon_0 > 0$ with the following property. For every $|\varepsilon| \le \varepsilon_0$, problem (7.25) has a unique analytic solution $(\boldsymbol{\mu}^{(1)}, \boldsymbol{\lambda}^{(1)}, \boldsymbol{\chi}^{(1)}) \in \mathcal{A}_{\sigma/4,0}$, $p^{(1)} \in \mathbb{C}$. This solution admits the estimate*

$$\|\boldsymbol{\mu}^{(1)}\|_{\sigma/4,0} + \|\boldsymbol{\lambda}^{(1)}\|_{\sigma/4,0} + \|\boldsymbol{\chi}^{(1)}\|_{\sigma/4,0} + |p^{(1)}| \le c. \qquad (8.29)$$

If $k = 0$, then this solution has the decomposition

$$\boldsymbol{\mu}^{(1)} = -\boldsymbol{\mu}_0 + \boldsymbol{\mu}_\varepsilon, \quad \boldsymbol{\lambda}^{(1)} = \mathcal{C}\mathbf{e}_1 - (\boldsymbol{\mu}_0 \cdot \mathbf{t}_0)\mathbf{e}_2 + \boldsymbol{\lambda}_\varepsilon. \qquad (8.30)$$

Here the the constant $\mathcal{C}$ and functions $\boldsymbol{\mu}_\varepsilon$, $\boldsymbol{\lambda}_\varepsilon$ admit the estimates

$$|\mathcal{C} - 1| \le c|\varepsilon|, \quad \|\boldsymbol{\mu}_\varepsilon\|_{\sigma/4,0} + \|\boldsymbol{\lambda}_\varepsilon\|_{\sigma/4,0} \le c|\varepsilon|\|\boldsymbol{\mu}_0|. \qquad (8.31)$$

The constant c is independent of α, k, and ε.

Proof. Notice that problem (7.26) is a particular case of problem (8.5) with the right hand sides $g_{1*} = 0$, $\mathbf{g}_2 = 0$, and $\gamma = 1$. Hence, the existence of a solution and estimate (8.29) is a straightforward consequence of Theorem 8.1. It remains to prove decompositiion (8.31).

Now our task is to justify decomposition (8.30). Let $\boldsymbol{\mu}_1$, $\boldsymbol{\lambda}_1$, and p_1 be given by Lemma 8.3. Set

$$\boldsymbol{\chi}_1 = \overline{\boldsymbol{\chi}_1} + \boldsymbol{\chi}_1^*,$$

Here $\boldsymbol{\chi}_1^*$ is a solution solution to the equations

$$\partial \boldsymbol{\chi}_1^* = \mathbf{S}\boldsymbol{\mu}_1 + \mathbf{T}^\top \boldsymbol{\lambda}_1, \quad \overline{\boldsymbol{\chi}_1^*} = 0. \qquad (8.32)$$

By virtue of (8.22), the mean value of the right hand side of equation (8.32) over the torus $\mathbb{T}^{n-1}$ is equal to zero. Hence this problem have an analytic periodic solution. The constant vector $\overline{\boldsymbol{\chi}_1}$ is defined by

$$\overline{\boldsymbol{\chi}_1} = -\overline{(\mathbf{V}^{-\top} - \mathbf{I})\boldsymbol{\chi}^*} \qquad (8.33)$$

Represent the solution $(\boldsymbol{\mu}^{(1)}, \boldsymbol{\lambda}^{(1)}, \boldsymbol{\chi}^{(1)}, p^{(1)})$ to problem (7.25) in the form

$$\boldsymbol{\mu}^{(1)} = \mathcal{C}\boldsymbol{\mu}_1 + \boldsymbol{\mu}, \quad \boldsymbol{\lambda}^{(1)} = \mathcal{C}\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}, \quad \boldsymbol{\chi}^{(1)} = \mathcal{C}\boldsymbol{\chi}_1 + \boldsymbol{\chi}, \quad p^{(1)} = \mathcal{C}p_1 + p, \quad (8.34)$$

where the constant $\mathcal{C}$ will be specified below. The calculations show that

$$\partial \boldsymbol{\mu}_1 = 0, \qquad (8.35a)$$

$$\mathbf{J}\partial \boldsymbol{\lambda}_1 + \boldsymbol{\Omega} \boldsymbol{\lambda}_1 + \mathbf{T}\boldsymbol{\mu}_1 + p_1 \mathbf{W}^\top \mathbf{e}_1 = p_1 (\mathbf{W}^\top - \mathbf{I})\mathbf{e}_1 \qquad (8.35b)$$

$$-\partial \boldsymbol{\chi}_1 + \mathbf{S}\boldsymbol{\mu}_1 + \mathbf{T}^\top \boldsymbol{\lambda}_1 = 0 \qquad (8.35c)$$

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Next, notice that $\mathbf{V}^{-\top} - \mathbf{I} = \mathbf{u}'$, which yields

$$\overline{\mathbf{V}^{-\top} - \mathbf{I}} = \overline{\mathbf{u}'} = 0, \quad \overline{(\mathbf{V}^{-\top} - \mathbf{I})\boldsymbol{\chi}_1} = \overline{(\mathbf{V}^{-\top} - \mathbf{I})\boldsymbol{\chi}_1^*}$$

It follows from this and (8.33) that

$$\overline{\{\mathbf{V}^{-\top}\boldsymbol{\chi}_1\}} = 0. \tag{8.35d}$$

Finally set

$$\mathcal{C} = \left\{ \overline{\{\mathbf{W}\boldsymbol{\lambda}_1\} \cdot \mathbf{e}_1} + \overline{\{\boldsymbol{\chi}_1 \cdot \nabla w_1\}} \right\}^{-1}, \tag{8.35e}$$

Substituting decomposition (8.34) in equations (7.25) and using relations (8.35) we conclude that the quantities $(\boldsymbol{\mu}, \boldsymbol{\lambda}, \boldsymbol{\chi}, p)$ in (8.34) satisfy equations (8.5) with the right hand sides

$$g_1 = \mathcal{C}p_1 w_1^*, \quad \boldsymbol{g}_2 = -\mathcal{C}p_1(\mathbf{W}^\top - \mathbf{I})\mathbf{e}_1, \quad \gamma = 0. \tag{8.36}$$

In order to apply Theorem (8.1) to the obtained problem (8.5), we have to estimate the the constant $\mathcal{C}$ and functions g_1, $\mathbf{g}_2$. In order to estimate $\mathcal{C}$, we substitute decomposition (8.23) into the right hand side of relation (8.35e). We get

$$\{\mathbf{W}\boldsymbol{\lambda}_1\} \cdot \mathbf{e}_1 = 1 + \boldsymbol{\lambda}_2 + (\{(\mathbf{W} - \mathbf{I})\boldsymbol{\lambda}_1\} \cdot \mathbf{e}_1).$$

Notice that

$$|\boldsymbol{\lambda}_2| \leq c|\boldsymbol{\mu}_0| \leq c|\varepsilon|, \quad |\boldsymbol{\lambda}_1| \leq c, \quad |\mathbf{W} - \mathbf{I}| \leq c|\varepsilon|,$$

which yields the estimate

$$|\overline{\{\mathbf{W}\boldsymbol{\lambda}_1\} \cdot \mathbf{e}_1} - 1| \leq c|\varepsilon| \tag{8.37}$$

Next, estimate (7.15) and estimates (7.25) imply the inequality $\|\mathbf{S}\boldsymbol{\mu}_1 + \mathbf{T}^\top\boldsymbol{\lambda}_1\|_{\sigma/2,0} \leq c$. Hence we can apply Lemma E.2 to equation (8.32) to obtain $\|\boldsymbol{\chi}_1^*\|_{\sigma/4,0} \leq c$. On the other hand, we have $|\nabla w_1 *| \leq c|\varepsilon|$. It follows that

$$|\overline{\boldsymbol{\chi}_1 \cdot \nabla w_1^*}| = |\overline{\boldsymbol{\chi}_1^* \cdot \nabla w_1^*}| \leq c|\varepsilon|. \tag{8.38}$$

Combining (8.37), (8.38) and recalling expression (8.35e) we arrive at the estimate

$$|\mathcal{C}^{-1} - 1| \leq c|\varepsilon|$$

Choosing $|\varepsilon| \leq \varepsilon_0$ sufficiently small we obtain the desired estimate (8.31) for $\mathcal{C}$. We are now in a position to estimate the functions g_1 and $\mathbf{g}_2$ in (8.36). It follows from estimate (8.24) for p_1 and estimate (8.31) for $\mathcal{C}$ that

$$\|g_1\|_{\sigma/2,0} \leq c|p_1|\|w_1^*\|_{\sigma/2,0} \leq c|\boldsymbol{\mu}_0||\varepsilon|, \quad \|g_2\|_{\sigma/2,0} \leq c|p_1|\|\mathbf{W}-\mathbf{I}\|_{\sigma/2,0} \leq c|\boldsymbol{\mu}_0||\varepsilon|$$

Applying Theorem 8.1 to the problem (8.5) with the right hand sides (8.36) we arrive at the estimate

$$\|\boldsymbol{\mu}\|_{\sigma/4,0} + \|\boldsymbol{\lambda}\|_{\sigma/4,0} \leq c|\varepsilon||\boldsymbol{\mu}_0|. \tag{8.39}$$

Finally set

$$\boldsymbol{\mu}_\varepsilon = \boldsymbol{\mu} + (\mathcal{C}-1)\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2, \quad \boldsymbol{\lambda}_\varepsilon = \boldsymbol{\lambda} - (\mathcal{C}-1)(\mathbf{t}_0 \cdot \boldsymbol{\mu}_0)\mathbf{e}_2 + \mathcal{C}\boldsymbol{\lambda}_2.$$

It remains to note that decomposition (8.30) follows from decompositions (8.23) and (8.34), and estimates (8.31) for $\boldsymbol{\mu}_\varepsilon$ and $\boldsymbol{\lambda}_\varepsilon$ follow from estimate (8.31) for $\mathcal{C}$, estimate (8.39) for $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$ and estimates (8.24) for $\boldsymbol{\mu}_i$ and $\boldsymbol{\lambda}_i$. $\qquad\square$

9 Quadratic form

Let $\boldsymbol{\mu}^{(i)}$, $\boldsymbol{\lambda}^{(i)}$, given by Theorems 8.2 and 8.4, be solutions to problems (7.25) and (7.26). We define the quadratic form $L = (L_{ij})$, $1 \leq i, j \leq 2$, as follows

$$L_{ij} = \int_{\mathbb{T}^{n-1}} \left\{ \mathbf{S}\boldsymbol{\mu}^{(i)} \cdot \boldsymbol{\mu}^{(j)} - (\mathbf{J}\boldsymbol{\partial}\boldsymbol{\lambda}^{(i)} + \boldsymbol{\Omega}\boldsymbol{\lambda}^{(i)}) \cdot \boldsymbol{\lambda}^{(j)} \right\} d\boldsymbol{\xi}. \tag{9.1}$$

This form plays the key role in our analysis of the bifurcation equation (4.7). As it will be shown in the next section, the derivatives of the action functional are represented in the terms of this form. The following theorem, which is the main result of this section, describe the properties of the quadratic form (9.1).

Theorem 9.1. *Under the assumptions of of Theorem 8.1, there exists $\varepsilon_0 > 0$ with the following property. For every $|\varepsilon| \leq \varepsilon_0$, the elements of the quadratic form L satisfy the inequalities*

$$|L_{12}| \leq c|\varepsilon|, \quad |L_{11}| \leq c,$$
$$c^{-1}|w_1^*|_{-1}^2 \leq |L_{22}| \leq c\|w_1^*\|_{\sigma/2,0}^2 \leq c|\varepsilon|^2. \tag{9.2}$$

If $k = 0$, then

$$|L_{11} - (2\pi)^{n-1}\mathbf{K}_0\boldsymbol{\mu}_0 \cdot \boldsymbol{\mu}_0| \leq c|\varepsilon||\boldsymbol{\mu}_0|^2, \quad |L_{12}| \leq c|\varepsilon||\boldsymbol{\mu}_0|. \qquad (9.3)$$

Here the strictly positive constant c is independent of α, k, and ε, the vector $\boldsymbol{\mu}_0$ is given by (8.21).

Proof. We begin with the observation that

$$|L_{ij}| \leq c(|\boldsymbol{\mu}^{(i)}|_0|\boldsymbol{\mu}^{(j)}|_0 + |\boldsymbol{\lambda}^{(i)}|_1|\boldsymbol{\lambda}^{(j)}|_0)$$
$$\leq c(\|\boldsymbol{\mu}^{(i)}\|_{\sigma/4,0}\|\boldsymbol{\mu}^{(j)}\|_{\sigma/4,0} + \|\boldsymbol{\lambda}^{(i)}\|_{\sigma/4,0}|\boldsymbol{\lambda}^{(j)}|_{\sigma/4,0}).$$

Therefore, estimates of $|L_{ij}|$ from above in (9.3) obviously follows from estimates (8.12) and (8.29) in Theorems 8.2 and 8.4. Let us estimate L_{22} from below. It follows from the expressions (2.6) and (2.9) for $\mathbf{J}$ and $\boldsymbol{\Omega}$ that

$$L_{22} = \int_{\mathbb{T}^{n-1}} \left\{ \mathbf{S}\boldsymbol{\mu}^{(2)} \cdot \boldsymbol{\mu}^{(2)} + k\lambda_1^{(2)^2} - \lambda_2^{(2)^2} - \lambda_1^{(2)}\boldsymbol{\partial}\lambda_2^{(2)} + \lambda_2^{(2)}\boldsymbol{\partial}\lambda_1^{(2)} \right\} d\boldsymbol{\xi}.$$

Integrating by parts gives

$$L_{22} = \int_{\mathbb{T}^{n-1}} \left\{ \mathbf{S}\boldsymbol{\mu}^{(2)} \cdot \boldsymbol{\mu}^{(2)} + k\lambda_1^{(2)^2} - \lambda_2^{(2)^2} + 2\lambda_2^{(2)}\boldsymbol{\partial}\lambda_1^{(2)} \right\} d\boldsymbol{\xi}. \qquad (9.4)$$

Next, multiplying both sides of equation (7.26b) by $\mathbf{e}_2$ we obtain

$$-\boldsymbol{\partial}\lambda_1^{(2)} + \lambda_2^{(2)} + \mathbf{t}_2 \cdot \boldsymbol{\mu}^{(2)} = -(p^{(2)} + w_1^*)\mathbf{W}^\top\mathbf{e}_1 \cdot \mathbf{e}_2.$$

It follows that

$$2\boldsymbol{\partial}\lambda_1^{(2)} - \lambda_2^{(2)} = \lambda_2^{(2)} + 2\mathbf{t}_2 \cdot \boldsymbol{\mu}^{(2)} + 2(p^{(2)} + w_1^*)\mathbf{W}^\top\mathbf{e}_1 \cdot \mathbf{e}_2.$$

Substituting this expression into (9.4) and noting that $\mathbf{W}^\top\mathbf{e}_1 \cdot \mathbf{e}_2 = W_{12}$ we arrive at the identity

$$L_{22} = \int_{\mathbb{T}^{n-1}} \left\{ \mathbf{S}\boldsymbol{\mu}^{(2)} \cdot \boldsymbol{\mu}^{(2)} + k\lambda_1^{(2)^2} + \lambda_2^{(2)^2} \right.$$
$$\left. + 2\lambda_2^{(2)}(\mathbf{t}_2 \cdot \boldsymbol{\mu}^{(2)} + 2(p^{(2)} + w_1^*))W_{12} \right\} d\boldsymbol{\xi}. \quad (9.5)$$

Introduce the denotation $\varkappa = W_{12}W_{11}^{-1}$. Multiplying both side of (7.26b) by $\varkappa\mathbf{e}_1$ we arrive at

$$-(p^{(2)} + w_1^*)W_{12} = -\varkappa(p^{(2)} + w_1^*)W_{11} = \varkappa(\boldsymbol{\partial}\lambda_2^{(2)} - k\lambda_1^{(2)} + \mathbf{t}_1 \cdot \boldsymbol{\mu}^{(2)})$$

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Thus we get

$$\int_{\mathbb{T}^{n-1}} 2\lambda_2^{(2)}(\mathbf{t}_2 \cdot \boldsymbol{\mu}^{(2)} + 2(p^{(2)} + w_1^*))W_{12}\, d\boldsymbol{\xi} =$$

$$\int_{\mathbb{T}^{n-1}} \{ 2\lambda_2^{(2)}(\mathbf{t}_2 \cdot \boldsymbol{\mu}^{(2)}) - \varkappa\lambda_2^{(2)}\boldsymbol{\partial}\lambda_2^{(2)} + \varkappa k\lambda_2^{(2)}\lambda_1^{(2)} - \varkappa\lambda_2^{(2)}(\mathbf{t}_1 \cdot \boldsymbol{\mu}^{(2)})\}\, d\boldsymbol{\xi} =$$

$$\int_{\mathbb{T}^{n-1}} \{ 2\lambda_2^{(2)}(\mathbf{t}_2 \cdot \boldsymbol{\mu}^{(2)}) + \frac{1}{2}\boldsymbol{\partial}\varkappa(\lambda_2^{(2)})^2 + \varkappa k\lambda_2^{(2)}\lambda_1^{(2)} - \varkappa\lambda_2^{(2)}(\mathbf{t}_1 \cdot \boldsymbol{\mu}^{(2)})\}\, d\boldsymbol{\xi} := \mathbf{I}_0.$$

Let us estimate the quantity $\mathbf{I}_0$. It follows from estimates (7.4) for the matrix $\mathbf{W}$ in Theorem 7.1 and estimate (7.15) for the matrix $\mathbf{T}^\top = [\mathbf{t}_1, \mathbf{t}_2]$ in Lemma 7.2 that

$$|\varkappa| + |\boldsymbol{\partial}\varkappa| \le c|\varepsilon|, \quad |\mathbf{t}_2| \le c, \quad |\mathbf{t}_1| \le c|\varepsilon|. \tag{9.6}$$

Thus we get

$$|\mathbf{I}_0| \le c\int_{\mathbb{T}^{n-1}} |\lambda_2^{(2)}||\boldsymbol{\mu}^{(2)}|d\boldsymbol{\xi} + c|\varepsilon|\int_{\mathbb{T}^{n-1}}(|\lambda_2^{(2)}|^2 + k|\lambda_1^{(2)}|^2)d\boldsymbol{\xi} \le$$

$$c|\varepsilon|\int_{\mathbb{T}^{n-1}}(|\lambda_2^{(2)}|^2 + k|\lambda_1^{(2)}|^2)d\boldsymbol{\xi} + |\varepsilon|^{-1}\int_{\mathbb{T}^{n-1}}|\boldsymbol{\mu}^{(2)}|^2 d\boldsymbol{\xi} \tag{9.7}$$

On the other hand, we have

$$L_{22} = \int_{\mathbb{T}^{n-1}} \{ \mathbf{S}\boldsymbol{\mu}^{(2)} \cdot \boldsymbol{\mu}^{(2)} + k\lambda_1^{(2)^2} + \lambda_2^{(2)^2} \} + \mathbf{I}_0.$$

It follows from this and (9.7) that

$$L_{22} \ge (1 - c|\varepsilon|)|\lambda_2^{(2)}|_0^2 + k|\lambda_1^{(2)}|^2 - c|\varepsilon|^{-1}|\boldsymbol{\mu}^{(2)}|_0^2$$

Applying estimates (8.12) and (8.13) in Theorem 8.2 we arrive at the inequality

$$L_{22} \ge c^{-1}(1 - c|\varepsilon|)|w_1^*|_{-1}^2 - c|\varepsilon||w_1^*|_{-1}^2.$$

Choosing ε sufficiently small we obtain the desired estimate (9.2). It remains to prove inequalities (9.3). Notice that for $k = 0$, the equality $\boldsymbol{\Omega}\mathbf{a} = a_2\mathbf{e}_2$ holds for every vector $\mathbf{a}$. Substituting decomposition (8.30) into the expression (9.1) we arrive at the identity

$$L_{11} = \int_{\mathbb{T}^{n-1}} \{ \mathbf{S}_0\boldsymbol{\mu}_0 \cdot \boldsymbol{\mu}_0 - (\boldsymbol{\mu}_0 \cdot \mathbf{t}_0)^2 \}\, d\boldsymbol{\xi} + \mathbf{I}_1, \tag{9.8}$$

where

$$\mathbf{I}_1 = \int_{\mathbb{T}^{n-1}} (\mathbf{S} - \mathbf{S}_0) \boldsymbol{\mu}_0 \cdot \boldsymbol{\mu}_0 \, d\boldsymbol{\xi} + \int_{\mathbb{T}^{n-1}} \mathbf{S} \boldsymbol{\mu}_\varepsilon \cdot (2\boldsymbol{\mu}_0 + \boldsymbol{\mu}_\varepsilon) \, d\boldsymbol{\xi}$$
$$- \int_{\mathbb{T}^{n-1}} \big\{ (\mathbf{J}\boldsymbol{\partial}\boldsymbol{\lambda}_\varepsilon + \boldsymbol{\Omega}\boldsymbol{\lambda}_\varepsilon) \cdot \boldsymbol{\lambda}_\varepsilon + 2(\boldsymbol{\mu}_0 \cdot \mathbf{t}_0)\lambda_{\varepsilon,2} \big\}, d\boldsymbol{\xi} \quad (9.9)$$

Estimates (7.15) for the matrices $\mathbf{S}$ and $\mathbf{T}^\top = [\mathbf{t}_1, \mathbf{t}_2]$ in Lemma 7.2 imply

$$|\mathbf{S} - \mathbf{S}_0| \le c|\varepsilon|, \quad |\mathbf{t}_2| \le c.$$

In its turn, estimates (8.31) in Theorem 8.4 imply

$$|\boldsymbol{\mu}_\varepsilon| + |\boldsymbol{\partial}\boldsymbol{\mu}_\varepsilon + ||\boldsymbol{\lambda}_\varepsilon| \le c|\varepsilon||\boldsymbol{\mu}_0|.$$

Combining these estimates we finally obtain

$$|\mathbf{I}_1| \le c|\varepsilon||\boldsymbol{\mu}_0|^2. \tag{9.10}$$

Next notice that

$$\int_{\mathbb{T}^{n-1}} \big\{ \mathbf{S}_0 \boldsymbol{\mu}_0 \cdot \boldsymbol{\mu}_0 - (\boldsymbol{\mu}_0 \cdot \mathbf{t}_0)^2 \big\} \, d\boldsymbol{\xi} = (2\pi)^{n-1} \mathbf{K}_0 \boldsymbol{\mu}_0 \cdot \boldsymbol{\mu}_0.$$

Substituting this equality into (9.8) and using inequality (9.10) we obtain estimate (9.3) for L_{11}. It remains to estimate L_{12}. Since $k = 0$ we have

$$L_{12} = \int_{\mathbb{T}^{n-1}} \big\{ \mathbf{S}\boldsymbol{\mu}^{(1)} \cdot \boldsymbol{\mu}^{(2)} - (\mathbf{J}\boldsymbol{\partial}\boldsymbol{\lambda}^{(2)} + \boldsymbol{\Omega}\boldsymbol{\lambda}^{(2)}) \cdot (\boldsymbol{\lambda}^{(1)} - \mathcal{C}\mathbf{e}_1) \big\} \, d\boldsymbol{\xi}. \tag{9.11}$$

In view of estimates (8.31) in Theorem 8.4, we have

$$|\boldsymbol{\mu}^{(1)}| \le c|\boldsymbol{\mu}_0, \quad |\boldsymbol{\lambda}^{(1)} - \mathcal{C}\mathbf{e}_1|| \le c|\boldsymbol{\mu}_0|, \quad |\boldsymbol{\partial}\boldsymbol{\lambda}^{(1)}| \le c|\boldsymbol{\mu}_0|$$

On the other hand, estimates (8.12) in Theorem 8.4 yield

$$\|\boldsymbol{\lambda}^{(2)}\|_{\sigma/4,0} + \|\boldsymbol{m}\boldsymbol{u}^{(2)}\|_{\sigma/4,0} + |p^{(2)}| \le c|\varepsilon|.$$

This result and the expression (9.11) for L_{12} lead to the estimate $|L_{12}| \le c|\varepsilon||\boldsymbol{\mu}_0|$. This completes the proof of Theorem $\qquad\square$

10   Action functional. Proof of Theorem 2.2

Theorem 7.1 constitutes the existence and uniqueness of the solution

$$\mathfrak{u} = \big(\boldsymbol{\varphi}(\alpha, k), e(\alpha, k)\big), m(\alpha, k), M(\alpha, k),$$
$$\boldsymbol{\varphi}(\alpha, k) = (\boldsymbol{\beta}, \varphi_0, \mathbf{u}, \mathbf{w}, W_{11}, W_{12}, W_{21}). \tag{10.1}$$

of the main operator equation (4.16) for every $(\alpha, k) \in \mathbb{T} \times [0, 1]$ and for all sufficiently small ε. This means that the vector

$$\boldsymbol{\Theta}(\boldsymbol{\varphi}) = \big(\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{V}, \boldsymbol{\Lambda}, \mathbf{W}, \mathbf{R}_i\big), \tag{10.2}$$

with the components $\mathbf{v}$, $\mathbf{V}$, $\mathbf{W}$, and $\mathbf{R}_i$ defined by relations (3.8), determines the canonical mapping $\boldsymbol{\vartheta}$ given by (3.1). By the definition of the operator Φ, this mapping puts the modified Hamiltonian H_m into the normal form (2.8). By virtue of Definition 2.1 of the normal form, the modified Hamiltonian has a weakly hyperbolic invariant torus. In the phase space, these torus has the the parametric representation

$$\mathbf{x}(\boldsymbol{\xi}) = \boldsymbol{\xi} + \mathbf{u}(\boldsymbol{\xi}), \quad \mathbf{y}(\boldsymbol{\xi}) = \mathbf{v}(\boldsymbol{\xi}), \quad \mathbf{z} = \mathbf{w}(\boldsymbol{\xi}), \quad \boldsymbol{\xi} \in \mathbb{T}^{n-1}, \tag{10.3}$$

in which $\mathbf{u}$, $\mathbf{w}$ are the components of the vector $\boldsymbol{\varphi}(\alpha, k)$, and the component $\mathbf{v}$ of the vector $\boldsymbol{\Theta}$ is defined by the relation (3.7), i.e.,

$$\mathbf{v} = \boldsymbol{\beta} + \mathbf{V}(\nabla \varphi_0 - w_2 \nabla w_1), \quad \mathbf{V} = (\mathbf{I} + \mathbf{u}')^{-\top}. \tag{10.4}$$

By virtue of Theorem 6.2, the main operator equation (4.16) has an analytic periodic solution for all $(\alpha, k) \in \mathbb{T} \times [0, 1]$ and for all sufficiently small ε. In other words, the modified Hamiltonian has the two-parametric family of weakly hyperbolic invariant tori labeled by (α, k). Notice that $H_m(\mathbf{x}, \mathbf{y}, \mathbf{z}) = H(\mathbf{x}, \mathbf{y}, \mathbf{z}) + mz_1 + 2^{-1}Mz_1^2$, where $m = m(\alpha, k)$, $M = M(\alpha, k)$ are the components of the vector $\mathfrak{u}$ satisfying (4.16). Therefore, the hamiltonian H has a weakly hyperbolic invariant torus if and only if

$$m(\alpha, k) = 0, M(\alpha, k) = 0 \tag{10.5}$$

Relations (10.5) give the system of two scalar bifurcation equations for $(\alpha, k) \in \mathbb{T}^1 \times [0, 1]$. The difficulty is that the scalars (m, M) are the integral part of a solution to the complicated operator equation, and we know nothing about their properties. In order to cope with this problem, we notice that every

Hamiltonian system has a variational formulation, and its solutions are critical points of the action functional. For quasi-periodic solutions, the action functional can be written as the Perceval functional, see [16],

$$\Psi = \int_{\mathbf{T}^{n-1}} \left((\boldsymbol{\omega} + \boldsymbol{\partial}\mathbf{u})\mathbf{v} + w_2 \boldsymbol{\partial} w_1 - H(\boldsymbol{\xi} + \mathbf{u}, \mathbf{v}, \mathbf{w}) \right) d\boldsymbol{\xi}. \qquad (10.6)$$

In view of Theorem 7.1, the mappings

$$\mathbb{R} \times [0,1] \ni (\alpha, k) \to \mathbf{u}(\alpha, k) \in \mathcal{A}_{\sigma/4,d},$$
$$\mathbb{R} \times [0,1] \ni (\alpha, k) \to \mathbf{v}(\alpha, k) \in \mathcal{A}_{\sigma/4,d},$$
$$\mathbb{R} \times [0,1] \ni (\alpha, k) \to \mathbf{w}(\alpha, k) - \alpha \mathbf{e}_1 \in \mathcal{A}_{\sigma/2,d},$$
$$\mathbb{R} \times [0,1] \ni (\alpha, k) \to \mathbf{W}(\alpha, k) \in \mathcal{A}_{\sigma/2,d}, \qquad (10.7)$$
$$\mathbb{R} \times [0,1] \ni (\alpha, k) \to M(\alpha, k) \in \mathbb{R}^1,$$
$$\mathbb{R} \times [0,1] \ni (\alpha, k) \to m(\alpha, k) + \alpha M(\alpha, k) \in \mathbb{R}^1,$$

are continuously differentiable and 2π-periodic in α. Moreover, they are analytic in $\mathbb{R} \times (0,1)$ and satisfy the inequalities

$$\|\partial_\alpha^r \mathbf{u}\|_{\sigma/2,d} + \|\partial_\alpha^r \mathbf{v}\|_{\sigma/2,d} \le c(r)|\varepsilon|,$$
$$\|\partial_\alpha^r (\mathbf{w} - \alpha \mathbf{e}_1)\|_{\sigma/2,d} + +\|\partial_\alpha^r (\mathbf{W} - \mathbf{I})\|_{\sigma/2,d} \le c(r)|\varepsilon|, \qquad (10.8)$$
$$|\partial_\alpha^r \boldsymbol{\beta}| + |\partial_\alpha^r (m + \alpha M)| + |\partial_\alpha^r (M + k)| \le c(r)|\varepsilon|,$$

and

$$\|\partial_\alpha^r \partial_k \mathbf{u}\|_{\sigma/2,d} + \|\partial_\alpha^r \partial_k \mathbf{v}\|_{\sigma/2,d} \le c|\varepsilon|,$$
$$\|\partial_\alpha^r \partial_k (\mathbf{w} - \alpha \mathbf{e}_1)\|_{\sigma/2,d} + \|\partial_\alpha^r \partial_k (\mathbf{W} - \mathbf{I})\|_{\sigma/2,d} \le c|\varepsilon|, \qquad (10.9)$$
$$|\partial_\alpha^r \partial_k (m + \alpha M)| + |\partial_\alpha^r \partial_k (M + k)| \le c(r)|\varepsilon|,$$

where $r \ge 0$ is an arbitrary integer, the constant c is independent of α, k, and ε. It follows from this that the mapping Ψ is a function of the variables (α, k), which belong to the Banach space $C^1([0,1]; \mathcal{A}_{\sigma/2,0})$. In particular, Ψ and $\partial_k \Psi$ are continuous in k and analytic in α "uniformly" with respect to $k \in [0,1]$.

The following Theorem constitutes the relations between Ψ, m, M and the quadratic form (L_{ij}).

Theorem 10.1. *Under the above assumptions, there is $\varepsilon_0 > 0$ with the properties. For every $|\varepsilon| \leq \varepsilon_0$ the derivatives of the function Ψ admit the representation*

$$\partial_\alpha \Psi(\alpha, k) = (2\pi)^{n-1}(m + \alpha M) + \varsigma_1 M, \tag{10.10}$$

$$\partial_k \Psi(\alpha, k) = M \, \partial_k M \, L_{22}, \tag{10.11}$$

$$\partial_\alpha^2 \Psi(\alpha, k) = (2\pi)^{n-1} M + \varsigma_2 M + L_{11} + 2L_{12}\partial_\alpha M + L_{22}(\partial_\alpha M)^2. \tag{10.12}$$

Here L_{ij} are given by (9.1), the quantities $\varsigma_i(\alpha, k)$ satisfy the inequalities

$$|\varsigma_i| + |\partial_\alpha \varsigma_i| \leq c|\varepsilon|^2. \tag{10.13}$$

Proof. Differentiation (10.6) with respect to the variable τ, $\tau = \alpha, k$, and integrating by parts gives

$$\partial_\tau \Psi(\alpha, k) = \int_{\mathbf{T}^{n-1}} \left((\boldsymbol{\omega} + \boldsymbol{\partial} \mathbf{u} - \nabla_y H(\boldsymbol{\xi} + \mathbf{u}, \mathbf{v}, \mathbf{w}) \right) \cdot \partial_\tau \mathbf{v} \, d\boldsymbol{\xi}$$

$$- \int_{\mathbf{T}^{n-1}} \left(\boldsymbol{\partial} \mathbf{v} + \nabla_x H(\boldsymbol{\xi} + \mathbf{u}, \mathbf{v}, \mathbf{w}) \right) \cdot \partial_\tau \mathbf{u} \, d\boldsymbol{\xi} +$$

$$\int_{\mathbf{T}^{n-1}} \left(\boldsymbol{\partial} w_1 - \partial_{z_2} H(\boldsymbol{\xi} + \mathbf{u}, \mathbf{v}, \mathbf{w}) \right) \cdot \partial_\tau w_2 \, d\boldsymbol{\xi}$$

$$- \int_{\mathbf{T}^{n-1}} \left(\boldsymbol{\partial} w_2 + \partial_{z_1} H(\boldsymbol{\xi} + \mathbf{u}, \mathbf{v}, \mathbf{w}) \right) \cdot \partial_\tau w_1 \, d\boldsymbol{\xi} \tag{10.14}$$

Next notice that

$$\nabla_y H = \nabla_y H_m, \quad \nabla_x H = \nabla_x H_m, \quad \partial_{z_2} H = \partial_{z_2} H_m,$$

$$\partial_{z_1} H(\boldsymbol{\xi} + \mathbf{u}, \mathbf{v}, \mathbf{w})) + m + M w_1 = \partial_{z_1} H_m(\boldsymbol{\xi} + \mathbf{u}, \mathbf{v}, \mathbf{w})). \tag{10.15}$$

Recall that the vector $\boldsymbol{\varphi}$ given by (10.1) serves as a solution to operator equation (4.16). From this and Lemma 4.1 we conclude that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ satisfy equations (4.11), i.e.,

$$\boldsymbol{\omega} + \boldsymbol{\partial} \mathbf{u} = \nabla_y H_m(\boldsymbol{\xi} + \mathbf{u}, \mathbf{v}, \mathbf{w}),$$

$$\boldsymbol{\partial} \mathbf{v} = -\nabla_x H_m(\boldsymbol{\xi} + \mathbf{u}, \mathbf{v}, \mathbf{w}), \quad \boldsymbol{\partial} \mathbf{w} = \mathbf{J} \nabla_z H_m(\boldsymbol{\xi} + \mathbf{u}, \mathbf{v}, \mathbf{w}).$$

Substituting these equality along with (10.15) into (10.14) we arrive at the identity

$$\partial_\tau \Psi(\alpha, k) = \int_{\mathbf{T}^{n-1}} (m + w_1 M)\partial_\tau w_1 \, d\boldsymbol{\xi}, \quad \tau = \alpha, k. \tag{10.16}$$

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Recall the denotations

$$\overline{w_1} = (2\pi)^{-n+1} \int_{\mathbb{T}^{n-1}} w_1 \, d\boldsymbol{\xi}, \quad w_1^* = w_1 - \overline{w_1}.$$

Notice that

$$w_1 = \alpha + w_1^*, \quad \partial_\alpha w_1 = 1 + \partial_\alpha w_1^*, \quad \partial_k w_1 = \partial_k w_1^*.$$

From this and (10.16) we obtain

$$\partial_\alpha \Psi(\alpha, k) = (2\pi)^{n-1}(m + \alpha M) + \varsigma_1 M,$$
$$\partial_k \Psi(\alpha, k) = M \int_{\mathbb{T}^{n-1}} w_1^* \partial_k w_1^* \, d\boldsymbol{\xi}, \tag{10.17}$$

where

$$\varsigma_1 = \int_{\mathbb{T}^{n-1}} w_1^* \partial_\alpha w_1^* \, d\boldsymbol{\xi}.$$

It obviously follows from estimates (10.8) that ς_1 satisfies inequality (10.13). This leads to representation (10.10).

Let us prove representation (10.11). Recall formulae (7.10) which constitutes the linear algebraic relation between the vector field $(\partial_\tau \mathbf{u}, \partial_\tau \mathbf{v}, \partial_\tau \mathbf{w})$ and the vector field $(\boldsymbol{\mu}^{(\tau)}, \boldsymbol{\lambda}^{(\tau)}, \boldsymbol{\chi}^{(\tau)})$. In view of the Second Structure Theorem (3.2) this relation has the inverse given by (3.19). In particular, we have

$$\partial_\tau \mathbf{w} = \mathbf{W} \boldsymbol{\lambda}^{(\tau)} + \chi_i^{(\tau)} \frac{\partial}{\partial \xi_i} \mathbf{w}. \tag{10.18}$$

On the other hand, relation (7.24) in Corollary 7.3 yields

$$(\boldsymbol{\mu}^{(k)}, \boldsymbol{\lambda}^{(k)}, \boldsymbol{\chi}^{(k)}) = \partial_k M(\boldsymbol{\mu}^{(2)}, \boldsymbol{\lambda}^{(2)}, \boldsymbol{\chi}^{(2)})$$

Substituting this relation into identity (10.18) with τ replaced by k, we obtain

$$\partial_k w_1^* = \partial_k \mathbf{w} \cdot \mathbf{e}_1 = \partial_k M \{ (\mathbf{W}^\top \mathbf{e}_1) \cdot \boldsymbol{\lambda}^{(2)} + \nabla w_1^* \cdot \boldsymbol{\chi}^{(2)} \}. \tag{10.19}$$

It follows that

$$\int_{\mathbb{T}^{n-1}} w_1^* \partial_k w_1^* \, d\boldsymbol{\xi} = \int_{\mathbb{T}^{n-1}} (p^{(2)} + w_1^*) \partial_k w_1^* \, d\boldsymbol{\xi} =$$

$$\partial_k M \int_{\mathbb{T}^{n-1}} \{ (p^{(2)} + w_1^*) \mathbf{W}^\top \mathbf{e}_1 \} \cdot \boldsymbol{\lambda}^{(2)} \, d\boldsymbol{\xi} + \partial_k M \int_{\mathbb{T}^{n-1}} \{ (p^{(2)} + w_1^*) \nabla w_1^* \} \cdot \boldsymbol{\chi}^{(2)} \, d\boldsymbol{\xi}$$

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Next, equations (7.26) imply the equalities

$$(p^{(2)} + w_1^*)\nabla w_1^* = -\boldsymbol{\partial}\,\boldsymbol{\mu}^{(2)},$$
$$(p^{(2)} + w_1^*)\mathbf{W}^\top\boldsymbol{e}_1 = -\mathbf{J}\boldsymbol{\partial}\,\boldsymbol{\lambda}^{(2)} - \boldsymbol{\Omega}\,\boldsymbol{\lambda}^{(2)} - \mathbf{T}\boldsymbol{\mu}^{(2)}.$$

Combining the obtained results we arrive at the identity

$$\int_{\mathbb{T}^{n-1}} w_1^*\partial_k w_1^*\,d\boldsymbol{\xi} = -\partial_k M \int_{\mathbb{T}^{n-1}} \Big((\mathbf{J}\boldsymbol{\partial}\,\boldsymbol{\lambda}^{(2)}$$
$$+ \boldsymbol{\Omega}\,\boldsymbol{\lambda}^{(2)} + \mathbf{T}\boldsymbol{\mu}^{(2)}) \cdot \boldsymbol{\lambda}^{(2)} + \boldsymbol{\partial}\,\boldsymbol{\mu}^{(2)} \cdot \boldsymbol{\chi}^{(2)}\Big)\,d\boldsymbol{\xi}. \quad (10.20)$$

Equation (7.26c) yields the identity

$$-\partial_k M \int_{\mathbb{T}^{n-1}} \boldsymbol{\partial}\,\boldsymbol{\mu}^{(2)} \cdot \boldsymbol{\chi}^{(2)}\,d\boldsymbol{\xi} = \partial_k M \int_{\mathbb{T}^{n-1}} \boldsymbol{\partial}\,\boldsymbol{\mu}^{(2)} \cdot \boldsymbol{\chi}^{(2)}\,d\boldsymbol{\xi}$$
$$= \partial_k M \int_{\mathbb{T}^{n-1}} \big(\mathbf{S}\boldsymbol{\mu}^{(2)} + \mathbf{T}^\top\boldsymbol{\lambda}^{(2)}\big) \cdot \boldsymbol{\mu}^{(2)}\,d\boldsymbol{\xi}$$

Substituting this equality into (10.20) we finally obtain

$$\int_{\mathbb{T}^{n-1}} w_1^*\partial_k w_1^*\,d\boldsymbol{\xi} = \partial_k M \int_{\mathbb{T}^{n-1}} \Big(\mathbf{S}\boldsymbol{\mu}^{(2)} \cdot \boldsymbol{\mu}^{(2)}$$
$$- (\mathbf{J}\boldsymbol{\partial}\,\boldsymbol{\lambda}^{(2)} + \boldsymbol{\Omega}\,\boldsymbol{\lambda}^{(2)}) \cdot \boldsymbol{\lambda}^{(2)}\Big)\,d\boldsymbol{\xi} = \partial_k M\ L_{22}. \quad (10.21)$$

It remains to notice that desired identity (10.11) obviously follows from (10.21) and identity (10.17).

Our next task is to prove identity (10.12). Differentiating identity (10.10) with respect to α and noting that $\partial_\alpha^2 w_1 = 0$ we obtain

$$\partial_\alpha^2 \Psi(\alpha, k) = M \int_{\mathbb{T}^{n-1}} \big((\partial_\alpha w_1)^2 + w_1^*\partial_\alpha^2 w_1^*\big)\,d\boldsymbol{\xi} + R, \quad (10.22)$$

where

$$R = \int_{\mathbb{T}^{n-1}} (\partial_\alpha m + w_1\partial_\alpha M)\,d\boldsymbol{\xi}. \quad (10.23)$$

Since $(\partial_\alpha w_1)^2 = 1 + 2\partial_\alpha w_1^* + (\partial_\alpha w_1^*)^2$, we have

$$\int_{\mathbb{T}^{n-1}} \big((\partial_\alpha w_1)^2 + w_1^*\partial_\alpha^2 w_1^*\big)\,d\boldsymbol{\xi} = (2\pi)^{n-1} + \varsigma_2, \quad (10.24)$$

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where

$$\varsigma_2 = \int_{\mathbb{T}^{n-1}} ((\partial_\alpha w_1^*)^2 + w_1^* \partial_\alpha^2 w_1^*) \, d\boldsymbol{\xi}. \tag{10.25}$$

Estimate (10.8) implies that ς_2 satisfies inequality (10.13). It remains to calculate R. Equality (7.11) yields

$$\partial_\alpha m + w_1 \partial_\alpha M = \partial_\alpha m + \alpha \partial_\alpha M + \partial_\alpha M w_1^* = p^{(\alpha)} + \partial_\alpha M w_1^*.$$

Thus we get

$$R = \int_{\mathbb{T}^{n-1}} (p^{(\alpha)} + \partial_\alpha M w_1^*) \partial_\alpha w_1 \, d\boldsymbol{\xi}. \tag{10.26}$$

Next, relation (10.18) with $\tau = \alpha$ gives the identity

$$\partial_\alpha w_1 = \partial_\alpha \mathbf{w} \cdot \mathbf{e}_1 = \mathbf{W}^\top \mathbf{e}_1 \cdot \boldsymbol{\lambda}^{(\alpha)} + \boldsymbol{\chi}^{(\alpha)} \cdot \nabla \mathbf{w}_1^*. \tag{10.27}$$

Substituting this result into (10.26) we arrive at the identity

$$R = \int_{\mathbb{T}^{n-1}} (p^{(\alpha)} + \partial_\alpha M w_1^*) \mathbf{W}^\top \mathbf{e}_1 \cdot \boldsymbol{\lambda}^{(\alpha)} \, d\boldsymbol{\xi}$$
$$+ \int_{\mathbb{T}^{n-1}} (p^{(\alpha)} + \partial_\alpha M w_1^*) \nabla w_1^* \cdot \boldsymbol{\chi}^{(\alpha)} \, d\boldsymbol{\xi}. \tag{10.28}$$

Next, equations (7.13a) and (7.13b) in Lemma 7.2 imply the equalities

$$(p^{(\alpha)} + w_1^*) \nabla w_1^* = -\partial \boldsymbol{\mu}^{(\alpha)},$$
$$(p^{(\alpha)} + w_1^*) \mathbf{W}^\top \mathbf{e}_1 = -\mathbf{J} \partial \boldsymbol{\lambda}^{(\alpha)} - \boldsymbol{\Omega} \boldsymbol{\lambda}^{(\alpha)} - \mathbf{T} \boldsymbol{\mu}^{(\alpha)},$$

Substituting these equalities into (10.26) we arrive at the identity

$$R = -\int_{\mathbb{T}^{n-1}} \left((\mathbf{J} \partial \boldsymbol{\lambda}^{(\alpha)} + \boldsymbol{\Omega} \boldsymbol{\lambda}^{(\alpha)} + \mathbf{T} \boldsymbol{\mu}^{(2)}) \cdot \boldsymbol{\lambda}^{(\alpha)} + \partial \boldsymbol{\mu}^{(\alpha)} \cdot \boldsymbol{\chi}^{(\alpha)} \right) d\boldsymbol{\xi}. \tag{10.29}$$

On the other hand, equation (7.13c) with $\tau = \alpha$ implies

$$-\int_{\mathbb{T}^{n-1}} \partial \boldsymbol{\mu}^{(\alpha)} \cdot \boldsymbol{\chi}^{(\alpha)} \, d\boldsymbol{\xi} = \int_{\mathbb{T}^{n-1}} \partial \boldsymbol{\mu}^{(\alpha)} \cdot \boldsymbol{\chi}^{(\alpha)} \, d\boldsymbol{\xi}$$
$$= \int_{\mathbb{T}^{n-1}} \left(\mathbf{S} \boldsymbol{\mu}^{(\alpha)} + \mathbf{T}^\top \boldsymbol{\lambda}^{(\alpha)} \right) \cdot \boldsymbol{\mu}^{(2)} \, d\boldsymbol{\xi}$$

Substituting this equality into (10.29) we obtain

$$R = \int_{\mathbb{T}^{n-1}} \left(\mathbf{S}\boldsymbol{\mu}^{(\alpha)} \cdot \boldsymbol{\mu}^{(\alpha)} - (\mathbf{J}\boldsymbol{\partial}\,\boldsymbol{\lambda}^{(\alpha)} + \boldsymbol{\Omega}\,\boldsymbol{\lambda}^{(\alpha)}) \cdot \boldsymbol{\lambda}^{(\alpha)} \right) d\boldsymbol{\xi} = L_{22}. \qquad (10.30)$$

Combining this relation with the equality

$$(\boldsymbol{\mu}^{(\alpha)}, \boldsymbol{\lambda}^{(\alpha)}) = (\boldsymbol{\mu}^{(1)}, \boldsymbol{\lambda}^{(1)}) + \partial_\alpha M\,(\boldsymbol{\mu}^{(2)}, \boldsymbol{\lambda}^{(2)})$$

and recalling formula (9.1) for L_{ij} we arrive at the expression for R,

$$R = L_{11} + \partial_\alpha M 2 L_{12} + (\partial_\alpha M)^2 L_{22},$$

Substituting this expression and equality (10.24) into (10.22) we obtain desired identity (10.13). This completes the proof of Theorem 10.1. $\qquad\square$

The following proposition is a direct consequence of this theorem.

Proposition 10.2. *Let all assumptions of Theorem 10.1 be satisfied and $k = 0$. Then the third derivative of Ψ admits the estimate*

$$|\partial_\alpha^3 \Psi(\alpha, 0) - (2\pi)^{n-1}\partial_\alpha M| \le c|\varepsilon|(|M| + |\boldsymbol{\mu}_0| + |\partial_\alpha M|). \qquad (10.31)$$

Proof. It follows from representation (10.12) in Theorem 10.1 that

$$\partial_\alpha^3 \Psi = (2\pi)^{n-1}\partial_\alpha M + \varsigma_2 \partial_\alpha M + \partial_\alpha \varsigma_2 M + \mathbf{I}_1 + \mathbf{I}_2, \qquad (10.32)$$

where

$$\mathbf{I}_1 = \partial_\alpha L_{11} + 2\partial_\alpha M \partial_\alpha L_{12} + (\partial_\alpha M)^2 \partial_\alpha L_{22}, \qquad (10.33)$$
$$\mathbf{I}_2 = 2\partial_\alpha^2 M L_{12} + 2(\partial_\alpha M)\partial_\alpha^2 M L_{22} \qquad (10.34)$$

The rest of the proof is based on the following

Lemma 10.3. *Under the assumptions of Theorem 10.2,*

$$|\partial_\alpha L_{11}| \le c|\varepsilon||\boldsymbol{\mu}_0|, \quad |\partial_\alpha L_{12}| + |\partial_\alpha L_{22}| \le c|\varepsilon|. \qquad (10.35)$$

Proof. First we estimates the vector fields $\boldsymbol{\mu}^{(i)}$ and $\boldsymbol{\lambda}^{(i)}$, $i = 1, 2$. We begin with the observation that relations in view of the identities (7.24),

$$(\boldsymbol{\mu}^{(2)}, \boldsymbol{\lambda}^{(2)}) = \partial_k M^{-1}\,(\boldsymbol{\mu}^{(k)}, \boldsymbol{\lambda}^{(k)}),$$
$$(\boldsymbol{\mu}^{(1)}, \boldsymbol{\lambda}^{(1)}) = (\boldsymbol{\mu}^{(\alpha)}, \boldsymbol{\lambda}^{(\alpha)}) - \partial_\alpha M\,(\boldsymbol{\mu}^{(2)}, \boldsymbol{\lambda}^{(2)}). \qquad (10.36)$$

In its turn, equalities (7.10) establish the following relations between the vector fields $\boldsymbol{\mu}^{(\tau)}$, $\boldsymbol{\lambda}^{(\tau)}$ and the derivatives of the vector fields $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, i.e.,

$$\boldsymbol{\chi}^{(\tau)} = \mathbf{V}^\top \partial_\tau \mathbf{u}$$

$$\boldsymbol{\lambda}^{(\tau)} = (W^{-1}\partial_\tau \mathbf{w} - \chi_i^{(\tau)} \mathbf{W}^{-1}\frac{\partial}{\partial \xi_i}\mathbf{w}$$

$$\boldsymbol{\mu}^{(\tau)} = \mathbf{V}^{-1}\Big(\partial_\tau \mathbf{v} + \chi_i^{(\tau)} \frac{\partial}{\partial \xi_i}\mathbf{v} - \boldsymbol{\Lambda}\boldsymbol{\lambda}^{(\tau)}\Big).$$

These identities along with estimates (10.8) and (10.9) imply

$$\|\partial_\alpha \boldsymbol{\mu}^{(\tau)}\|_{\sigma/4,0} + \|\partial_\alpha \boldsymbol{\lambda}^{(\tau)}\|_{\sigma/4,0} \le c|\varepsilon|, \quad \tau = \alpha, k. \tag{10.37}$$

On the other hand, inequalities (10.9) yield

$$|\partial_k M^{-1}| \le (1 - c|\varepsilon|)^{-1} \le c, \quad |\partial_k \partial_\alpha M| \le c|\varepsilon|. \tag{10.38}$$

Inequalities (10.37)-(10.38) and the identity

$$\partial_\alpha (\boldsymbol{\mu}^{(2)}, \boldsymbol{\lambda}^{(2)}) = -(\partial_k M)^{-2}\partial_\alpha \partial_k M \ (\boldsymbol{\mu}^{(k)}, \boldsymbol{\lambda}^{(k)}) + \partial_k M^{-1}\partial_\alpha (\boldsymbol{\mu}^{(k)}, \boldsymbol{\lambda}^{(k)})$$

lead to the estimate

$$|\partial_\alpha (\boldsymbol{\mu}^{(2)}, \boldsymbol{\lambda}^{(2)})| \le c|\varepsilon|. \tag{10.39}$$

Next, it follows from (10.36) that

$$\partial_\alpha (\boldsymbol{\mu}^{(1)}, \boldsymbol{\lambda}^{(1)}) = \partial_\alpha (\boldsymbol{\mu}^{(\alpha)}, \boldsymbol{\lambda}^{(\alpha)}) - \partial_\alpha^2 M \ (\boldsymbol{\mu}^{(2)}, \boldsymbol{\lambda}^{(2)})$$
$$- \partial_\alpha M \partial_\alpha (\boldsymbol{\mu}^{(2)}, \boldsymbol{\lambda}^{(2)}).$$

This relation along with inequalities (10.9) and (10.39) yields the estimate

$$|\partial_\alpha (\boldsymbol{\mu}^{(1)}, \boldsymbol{\lambda}^{(1)})| \le c|\varepsilon|. \tag{10.40}$$

The differentiation both sides of equality (9.1) with respect to α leads to the representation

$$\partial_\alpha L_{ij} = \sigma_{ij} + \int_{\mathbb{T}^{n-1}} \big\{ \ \mathbf{S}\mu^{(i)} \cdot \partial_\alpha \boldsymbol{\mu}^{(j)} - (\mathbf{J}\partial\boldsymbol{\lambda}^{(i)} + \boldsymbol{\Omega}\boldsymbol{\lambda}^{(i)}) \cdot \partial_\alpha \boldsymbol{\lambda}^{(j)} \ \big\} \, d\boldsymbol{\xi}$$

$$+ . \int_{\mathbb{T}^{n-1}} \big\{ \ \mathbf{S}\partial_\alpha \boldsymbol{\mu}^{(i)} \cdot \boldsymbol{\mu}^{(j)} - (\mathbf{J}\partial\boldsymbol{\lambda}^{(j)} + \boldsymbol{\Omega}\boldsymbol{\lambda}^{(j)}) \cdot \partial_\alpha \boldsymbol{\lambda}^{(i)} \ \big\} \, d\boldsymbol{\xi}, \tag{10.41}$$

where
$$\sigma_{ij} = \int_{\mathbb{T}^{n-1}} \partial_\alpha \mathbf{S}\boldsymbol{\mu}^{(i)} \cdot \boldsymbol{\mu}^{(i)} \, d\boldsymbol{\xi}.$$

Estimate (8.12) in Theorem 8.2 and estimate (8.31) in Theorem8.4 yield the inequality. that

$$|\sigma_{12}| + |\sigma_{22}| \le c|\boldsymbol{\mu}^{(1)}|_0 |\boldsymbol{\mu}^{(2)}|_0 + |\boldsymbol{\mu}^{(2)}|_0^2 \le c|\varepsilon|^2, \quad |\sigma_{11}| \le c|\boldsymbol{\mu}^{(1)}|_0^2 \le c|\varepsilon||\boldsymbol{\mu}_0|. \tag{10.42}$$

On the other hand, we have

$$\left| \int_{\mathbb{T}^{n-1}} \left\{ \mathbf{S}\mu^{(i)} \cdot \partial_\alpha \boldsymbol{\mu}^{(j)} - (\mathbf{J}\boldsymbol{\partial}\boldsymbol{\lambda}^{(i)} + \boldsymbol{\Omega}\boldsymbol{\lambda}^{(i)}) \cdot \partial_\alpha \boldsymbol{\lambda}^{(j)} \right\} d\boldsymbol{\xi} \right|$$
$$\le c(|\boldsymbol{\mu}^{(i)}|_0 |\partial_\alpha \boldsymbol{\mu}^{(j)}|_0 + |\boldsymbol{\lambda}^{(i)}|_1 |\partial_\alpha \boldsymbol{\lambda}^{(j)}|_0).$$

Combining this inequality with estimates (10.39)-(10.40) we arrive at the estimate

$$\left| \int_{\mathbb{T}^{n-1}} \left\{ \mathbf{S}\mu^{(i)} \cdot \partial_\alpha \boldsymbol{\mu}^{(j)} - (\mathbf{J}\boldsymbol{\partial}\boldsymbol{\lambda}^{(i)} + \boldsymbol{\Omega}\boldsymbol{\lambda}^{(i)}) \cdot \partial_\alpha \boldsymbol{\lambda}^{(j)} \right\} d\boldsymbol{\xi} \right|$$
$$\le c|\varepsilon|(|\boldsymbol{\mu}^{(i)}|_0 + |\boldsymbol{\lambda}^{(i)}|_1) \le c|\varepsilon|,$$

which along with (10.42) yields the estimate (10.35) for $\partial_\alpha L_{12}$ and $\partial_\alpha L_{22}$.

It remains to estimate $\partial_\alpha L_{11}$. Since $k = 0$, we have

$$\mathbf{J}\boldsymbol{\partial}\boldsymbol{\lambda}^{(1)} + \boldsymbol{\Omega}\boldsymbol{\lambda}^{(1)} = \mathbf{J}\boldsymbol{\partial}\boldsymbol{\lambda}^{(1)} + \boldsymbol{\lambda}^{(1)} \cdot \mathbf{e}_2.$$

Substituting decomposition (8.30) into this relation and using estimate (8.31) we obtain

$$|\mathbf{J}\boldsymbol{\partial}\boldsymbol{\lambda}^{(1)} + \boldsymbol{\Omega}\boldsymbol{\lambda}^{(1)}| = |\mathbf{J}\boldsymbol{\partial}\boldsymbol{\lambda}_\varepsilon + (\boldsymbol{\mu}_0 \cdot \mathbf{t}_0) + \boldsymbol{\lambda}_\varepsilon \cdot \mathbf{e}_2| \le c|\boldsymbol{\mu}_0|.$$

Combining this result with the estimate $|\boldsymbol{\mu}^{(1)}| \le c|\boldsymbol{\mu}_0|$ and estimates (10.39)-(10.40) we arrive at the inequality

$$\left| \int_{\mathbb{T}^{n-1}} \left\{ \mathbf{S}\mu^{(1)} \cdot \partial_\alpha \boldsymbol{\mu}^{(1)} - (\mathbf{J}\boldsymbol{\partial}\boldsymbol{\lambda}^{(1)} + \boldsymbol{\Omega}\boldsymbol{\lambda}^{(1)}) \cdot \partial_\alpha \boldsymbol{\lambda}^{(1)} \right\} d\boldsymbol{\xi} \right| \le c|\varepsilon||\boldsymbol{\mu}_0|.$$

Combing this inequality with estimate (10.42) for σ_{11} and identity (10.41) we finally obtain that $|\partial_\alpha L_{11}| \le c|\varepsilon| \, |\boldsymbol{\mu}_0|$. This completes the proof of the lemma. $\square$

Let us turn to the proof of Proposition 10.3. Applying Lemma 10.35 we obtain that the quantity $\mathbf{I}_1$ given by (10.33) satisfies the inequality

$$|\mathbf{I}_1| \leq c|\partial_\alpha L_{11}| + c|\partial_\alpha M|(|\partial_\alpha L_{12}| + |\partial_\alpha L_{22}|) \leq c|\varepsilon||\boldsymbol{\mu}_0| + c|\varepsilon||\partial_\alpha M|. \quad (10.43)$$

On the other hand, estimates (9.2) and (9.3) in Theorem 9.4 imply the following estimate for the quantity $\boldsymbol{I}_2$ given by (10.34)

$$|\mathbf{I}_2| \leq c|L_{12}| + c|\partial_\alpha M||L_{22}| \leq c|\varepsilon||\boldsymbol{\mu}_0| + c|\varepsilon||\partial_\alpha M|. \quad (10.44)$$

Substituting (10.43) and (10.44) into identity (10.32) we arrive at the estimate

$$|\partial_\alpha^3 \Psi(\alpha, 0) - (2\pi)^{n-1}\partial_\alpha M|$$
$$\leq c|\varepsilon||\varsigma_2||\partial_\alpha M| + c|\varepsilon||\partial_\alpha \varsigma_2||M| + c|\varepsilon|(|\partial_\alpha M| + |\boldsymbol{\mu}_0|)$$

Recalling estimate (10.13) in Theorem 10.1 for the quantity ς_2 we finally obtain the desired inequality

$$|\partial_\alpha^3 \Psi(\alpha, 0) - (2\pi)^{n-1}\partial_\alpha M| \leq c|\varepsilon|(|M| + |\partial_\alpha M| + |\boldsymbol{\mu}_0|),$$

and the proposition follows.

$\square$

10.1 Proof of Theorem 2.2

We are now in a position to prove Theorem 2.2 which is the main result of this work. We begin with the observation that by virtue of Theorem 6.2 the main operator equation (4.16) has an analytic periodic solution $(\boldsymbol{\varphi}, e, m, M)$ for all $(\alpha, k) \in \mathbb{T}^1 \times [0, 1]$ and all $|\varepsilon| \leq \varepsilon_0$. This solution define the vector $\boldsymbol{\Theta}(\varphi) = (\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{V}, \mathbf{W}, \boldsymbol{\Lambda}, \mathbf{R})$ such that the corresponding canonical mapping $\boldsymbol{\vartheta}$ put the modified Hamiltonian $H_m = H + mz_1 + 2^{-1}Mz_1^2$ into the normal form (2.8). In particular, the modified Hamiltonian has the weakly hyperbolic invariant torus for every $(\alpha, k) \in \mathbb{T}^1 \times [0, 1]$. Recall that m and M are functions of the variables α and k. Hence Theorem 2.2 will be proved if we prove that the bifurcation equations

$$m(\alpha, k) = M(\alpha, k) = 0 \quad (10.45)$$

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have a solution $(\alpha_0, k_0) \in \mathbb{T}^1 \times [0, 1]$. We claim that such a solution can be defined as a minimizer of the action functional $\Psi(\alpha, k)$ given by (10.6), i.e.,

$$\Psi(\alpha_0, k_0) = \min_{(\alpha,k) \in \mathbb{T}^1 \times [0,1]} \Psi(\alpha, k). \qquad (10.46)$$

The proof of this fact falls into the sequence of lemmas. The first lemma shows that every minimizer (α_0, k_0), $k_0 > 0$, of the action functional serves as a solution to the bifurcation equations.

Lemma 10.4. *Let all assumptions of Theorem (10.1) be satisfied. Furthermore, assume that*

$$\Psi(\alpha_0, k_0) = \min_{(\alpha,k) \in \mathbb{T}^1 \times [0,1]} \Psi(\alpha, k) \quad and \quad \Psi(\alpha_0, 0) > \Psi(\alpha_0, k_0). \qquad (10.47)$$

Then $m(\alpha_0, k_0) = M(\alpha_0, k_0) = 0$.

Proof. Let us prove that $L_{22}(\alpha_0, k) > 0$ everywhere on $(0, 1)$ but countable discrete set. Notice that the mapping $[0, 1] \ni k \to w_1^*(\alpha_0, k) \in \mathcal{A}_{\sigma/2,0}$ is continuously differentiable on $[0, 1]$ an is analytic on $(0, 1)$. If w_1^* is not identically equal to zero on $(0, 1)$, then there is a countable discrete set $\{k_i\} \subset (0, 1)$ such that $|w_1^*|_{-1} > 0$ outside of this set. It follows from this and estimate (9.2) in Theorem 9.1 that $L_{22}(\alpha_0, k) > 0$ for $k \neq k_i$.

Let us show that the case when $w_1^*(\alpha_0, \cdot)$ identically equals zero on $(0,1)$ is impossible. In view of (9.2), the identity $w_1^*(\alpha_0, \cdot) \equiv 0$ on $(0, 1)$ implies the identity $L_{22}(\alpha_0, \cdot) \equiv 0$ on $(0, 1)$, and hence implies the identity $\partial_k \Psi(\alpha_0, \cdot) = M \partial_k M L_{22}(\alpha_0, \cdot) \equiv 0$ on $(0, 1)$. The latter is impossible since $\Psi(\alpha_0, 0) > \Psi(\alpha_0, k_0)$. Therefore, $L_{22}(\alpha_0, k) > 0$ for all $k \neq k_i$.

Let us prove that $k_0 < 1$. It follows from inequalities (10.9) and (10.1) that

$$|\partial_k M + 1| + |M + k| \leq c|\varepsilon|.$$

Hence for all sufficiently small ε we have

$$\partial_k M \leq -1/3, \quad M < -1/3 \ \text{ for } \ k \geq 1/2.$$

Since $L_{22}(\alpha_0, k)$ is strictly positive a.e. in $(0, 1)$, it follows that $\Psi(\alpha_0, k)$ strictly increases on $(1/2, 1]$. Hence $k_0 < 1$. From this and the conditions of the lemma we conclude that $0 < k_0 < 1$.

Let us prove that $M(\alpha_0, k_0) = 0$. Notice that for all sufficiently small $|\varepsilon|$, we have $\partial_k M \geq -1 + c|\varepsilon| < 0$. Hence $\partial_k M L_{22}(\alpha_0, k) < 0$ almost everywhere on $(0, 1)$. Next, identity (10.11) in Theorem 10.1 implies $\partial_k \Psi(\alpha_0, k) =$

$M\partial_k M L_{22}(\alpha_0, k)$. If $M(\alpha_0, k_0) \neq 0$, then the function $\Psi(\alpha_0, k)$ is strictly monotone in a neighborhood of k_0. This contradicts to the assumption that (α_0, k_0) is a minimizer of Ψ. Hence $M(\alpha_0, k_0)$ equals zero.

It remains to prove that $m(\alpha_0, k_0) = 0$. To this end, notice that identity (10.10) in Theorem 10.1 along with the equality $M(\alpha_0, k_0) = 0$ yields

$$0 = \partial_\alpha \Psi(\alpha_0, k_0) = (2\pi)^{n-1}(m + \alpha M) + \varsigma_1 M = (2\pi)^{n-1} m(\alpha_0, k_0),$$

which completes the proof. □

Lemma 10.5. *Let all assumptions of Theorem* (10.1) *be satisfied. Furthermore, assume that*

$$\Psi(\alpha_0, 0) = \min_{(\alpha, k) \in \mathbb{T}^1 \times [0,1]} \Psi(\alpha, k) \quad and \quad \mathbf{K}_0 \boldsymbol{\mu}_0 \cdot \boldsymbol{\mu}_0 < 0. \tag{10.48}$$

Then $m(\alpha_0, 0) = M(\alpha_0, 0) = 0$.

Proof. We split the proof into two steps.
Step 1. First we prove the lemma under the additions assumptions that

$$\|w_1^*(\alpha_0, k)\|_{\sigma/2, 0} > 0 \quad \text{for a.e.} \quad k \in (0, 1). \tag{10.49}$$

Let us prove that $M(\alpha_0, 0) = 0$. Suppose, contrary to our claim, that $M(\alpha_0, 0) \neq 0$. Notice that $\partial_k M \leq -1 + c|\varepsilon| < 0$. On the other hand, estimate (9.2) in Theorem 9.1 yields $L_{22} \geq c^{-1}|w_1^*|_{-1}$. Hence $\partial_k M L_{22}(\alpha_0, k) < 0$ almost everywhere on $(0, 1)$. If $M(\alpha_0, 0) > 0$, then $\Psi(\alpha_0, k)$ strictly decreases in a neighborhood of $k = 0$. This contradicts to the assumptions that $(\alpha_0, 0)$ is a minimizer of Ψ. Hence $M(\alpha_0, 0) \leq 0$. Next, representation (10.12) in Theorem 10.1 yields the inequality

$$0 \leq \partial_\alpha^2 \Psi(\alpha_0, 0) = (2\pi)^{n-1} M(\alpha_0, 0) + \varsigma_2 M(\alpha_0, 0)$$
$$+ L_{11}(\alpha_0, 0) + 2L_{12}\partial_\alpha M(\alpha_0, 0) + L_{22}\partial_\alpha M(\alpha_0, 0)^2 \tag{10.50}$$

In view of Theorem 10.1, we have $|\varsigma_2| \leq c|\varepsilon|$. Hence the inequality

$$|M(\alpha_0, 0)| = -M(\alpha_0, 0) \leq L_{11}(\alpha_0, 0) + 2L_{12}\partial_\alpha M(\alpha_0, 0) + L_{22}\partial_\alpha M(\alpha_0, 0)^2 \tag{10.51}$$

holds true for all sufficiently small ε. Next, estimate (9.3) in Theorem (9.1) implies the inequality

$$L_{11} + 2|\partial_\alpha M||L_{12}| \leq (2\pi)^{n-1} \mathbf{K}_0 \boldsymbol{\mu}_0 \cdot \boldsymbol{\mu}_0 + c|\varepsilon||\boldsymbol{\mu}_0|^2 + c|\varepsilon||\boldsymbol{\mu}_0||\partial_\alpha M|$$
$$\leq (2\pi)^{n-1} \mathbf{K}_0 \boldsymbol{\mu}_0 \cdot \boldsymbol{\mu}_0 + c|\varepsilon||\boldsymbol{\mu}_0|^2 + c|\varepsilon||\partial_\alpha M|^2.$$

Recall that
$$-c^{-1}|\boldsymbol{\mu}_0|^2 \leq |K_0\boldsymbol{\mu}_0 \cdot \boldsymbol{\mu}_0| \leq -c|\boldsymbol{\mu}_0|^2$$
for some positive constant c independent of ε. From this we conclude that
$$L_{11} + 2|\partial_\alpha M||L_{12}| \leq \mathbf{K}_0\boldsymbol{\mu}_0 \cdot \boldsymbol{\mu}_0 + c|\varepsilon||\partial_\alpha M|^2 \tag{10.52}$$
for all sufficiently small ε. Substituting this inequality into (10.51) and noting that $L_{22} \leq c\varepsilon^2$ we conclude that
$$|M(\alpha_0, 0)| - \mathbf{K}_0\boldsymbol{\mu}_0 \cdot \boldsymbol{\mu}_0 \leq c|\varepsilon||\partial_\alpha M|^2,$$
which gives
$$|M(\alpha_0, 0)| + |\boldsymbol{\mu}_0|^2 \leq c|\varepsilon||\partial_\alpha M|^2, \tag{10.53}$$
It follows from representation (10.50) and estimates (9.2), (9.3) in Theorem 9.1 that
$$0 \leq \partial_\alpha^2 \Psi(\alpha_0, 0) \leq c|M(\alpha_0, 0)| + |L_{11}(\alpha_0, 0)| +$$
$$2|L_{12}||\partial_\alpha M(\alpha_0, 0)| + |L_{22}|\partial_\alpha M(\alpha_0, 0)^2 \leq c|M(\alpha_0, 0)| + c|\boldsymbol{\mu}_0|^2$$
$$+ c|\varepsilon||\boldsymbol{\mu}_0||\partial_\alpha M(\alpha_0, 0)| + c|\varepsilon||\partial_\alpha M(\alpha_0, 0)|^2$$

Substituting estimate (10.53) into the right hand side of this inequality we obtain
$$0 \leq \partial_\alpha^2 \Psi(\alpha_0, 0) \leq c|\varepsilon| |\partial_\alpha M(\alpha_0, 0)|^2. \tag{10.54}$$
Let us estimate the third derivatives of Ψ. Combining inequality (10.31) in Proposition 10.2 and inequality (10.53) we obtain
$$|\partial_\alpha^3 \Psi(\alpha_0, 0) - (2\pi)^{n-1}\partial_\alpha M(\alpha_0, 0)| \leq c|\varepsilon||\partial_\alpha M(\alpha_0, 0)|.$$
It follows that the inequality
$$|\partial_\alpha^3 \Psi(\alpha_0, 0)| \geq |\partial_\alpha M(\alpha_0, 0)| \tag{10.55}$$
holds for all small ε. Since $\Psi_0(\alpha_0, 0)$ is analytic and takes the minimum at point α_0 we have the Taylor expansion
$$0 \leq \Psi(\alpha_0 + t, 0) - \Psi(\alpha_0, 0) = At^2 + Bt^3 + Ct^4, \tag{10.56}$$
where
$$A = 2^{-1}\partial_\alpha^2 \Psi(\alpha_0, 0), \quad B = 6^{-1}\partial_\alpha^2 \Psi(\alpha_0, 0), \quad , |C| \leq c_4,$$

where c_4 is independent of ε. Notice that this relation holds true for all $t \in \mathbb{R}^1$. Estimates (10.54) and (10.55) imply the inequalities

$$|A| \leq c_2 |\varepsilon| |\partial_\alpha M|^2, \quad |B| \geq 6^{-1} |\partial_\alpha M|.$$

Now set $t = -\delta B$, where δ are an arbitrary positive number. We have

$$0 \leq \Psi(\alpha_0 + t, 0) - \Psi(\alpha_0, 0) = At^2 + Bt^3 + Ct^4 =$$
$$B^4 \delta^3 (AB^{-2}\delta^{-1} - 1 + \delta C) \quad (10.57)$$

Notice that $|AB^{-2}| \leq c|\varepsilon|$. Obviously, the right hand side of (10.57) is negative for $\delta = |\varepsilon|^{1/2}$ and small ε. This contradiction proves the equality $M(\alpha_0, 0) = 0$. It remains to note that the equality $m(\alpha_0, k) = 0$ obviously follows from the relations

$$0 = \partial_\alpha \Psi(\alpha_0, 0) = (2\pi)^{n-1}(m + \alpha M) + \varsigma_1 M = (2\pi)^{n-1} m(\alpha_0, 0).$$

Step 2. It remains to consider the degenerate case when $w_1^*(\alpha_0, k) = 0$ for all $k \in [0, 1]$. In view of estimate (8.12) we have in this case

$$\boldsymbol{\lambda}^{(2)}(\alpha_0, k) = 0, \quad \boldsymbol{\mu}^{(2)}(\alpha_0, k) = 0 \quad \text{for all} \quad k \in [0, 1].$$

Recalling representation (9.1) for L_{ij} we conclude that

$$L_{12}(\alpha_0, k) = 0, \quad L_{22}(\alpha_0, k) = 0 \quad \text{for all} \quad k \in [0, 1]. \quad (10.58)$$

It follows from this and formula (10.11) that $\partial_k \Psi(\alpha_0, k) = 0$ and hence

$$\Psi(\alpha_0, k) = \min_{(\alpha, l) \in \mathbb{T}^1 \times [0,1]} \Psi(\alpha, l) \quad \text{for all} \quad k \in [0, 1]. \quad (10.59)$$

In other words, Ψ, the whole segment $\{\alpha_0\} \times [0, 1]$ consists of the minimizers of Ψ.

Let us prove that there is $k_0 \in [0, 1]$ such that $M(\alpha_0, k_0) = 0$. We begin with the observation that $|M(\alpha_0, 1) + 1| \leq c|\varepsilon|$. Hence, $M(\alpha_0, 1) < 0$ for all sufficiently small ε. Therefore, it suffices to prove that $M(\alpha_0, 0) \geq 0$. Suppose, contrary to our claim, that $M(\alpha_0, 0) < 0$ Since $(\alpha_0, 0)$ is a minimizer of Ψ and $L_{12} = L_{22} = 0$ for $\alpha = \alpha_0$, representation (10.12) in Theorem 10.1 implies the inequality

$$0 \leq \partial_\alpha^2 \Psi(\alpha_0, 0) = (2\pi)^{n-1} M(\alpha_0, 0) + \varsigma_2 M(\alpha_0, 0) + L_{11}(\alpha_0, 0).$$

It follows from this, estimate (9.3) in Theorem 9.1, and the inequality $|\varsigma_2| \leq c|\varepsilon|$ that

$$0 \leq \partial_\alpha^2 \Psi(\alpha_0, 0) = (2\pi)^{n-1} M(\alpha_0, 0) +$$
$$\mathbf{K}_0 \boldsymbol{\mu}_0 \cdot \boldsymbol{\mu}_0 + c|\varepsilon| |M(\alpha_0, 0)| + c|\varepsilon| |\boldsymbol{\mu}_0|^2. \quad (10.60)$$

Recall that $M(\alpha_0, 0) < 0$ and $\mathbf{K}_0 \boldsymbol{\mu}_0 \cdot \boldsymbol{\mu}_0 \leq -c|\boldsymbol{\mu}_0|^2$. It follows from this that the right hand side of (10.60) is negative for all sufficiently small ε. The contradiction prove the inequality $M(\alpha_0, 0) \geq 0$. Since $M(\alpha_0, 1)$ is negative, there is k_0 such that $M(\alpha_0, k_0) = 0$. On the other hand, Ψ takes the minimum at the point (α_0, k_0). It follows from this that $\partial_\alpha \Psi(\alpha_0, k_0) = (2\pi)^{n-1} m(\alpha_0, k_0) = 0$. This completes the proof of the lemma. $\qquad \square$

It remains to note that the statement of Theorem 2.2 is a straightforward consequence of Lemmas 10.4 and 10.5.

A   Proof of Theorem 3.1

Let $\boldsymbol{\vartheta}$ is defined by (3.1). It follows that

$$\boldsymbol{\vartheta}' = \begin{pmatrix} A_{11} & 0 & 0 \\ A_{21} & A_{22} & A_{23} \\ A_{31} & 0 & A_{33} \end{pmatrix}, \quad \boldsymbol{\vartheta}'^\top = \begin{pmatrix} A_{11}^\top & A_{21}^\top & A_{31}^\top \\ 0 & A_{22}^\top & 0 \\ 0 & A_{23}^\top & A_{33}^\top \end{pmatrix}, \quad (1\ A)$$

where

$$A_{11} = \mathbf{I} + \mathbf{u}', \quad A_{22} = \mathbf{V}, \quad A_{33} = \mathbf{W},$$
$$A_{21} = \mathbf{v}'_\xi + (\mathbf{V}\boldsymbol{\eta})'_\xi + (\boldsymbol{\Lambda}\boldsymbol{\zeta})'_\xi + \frac{1}{2}(\boldsymbol{\zeta}^\top \mathbf{R}\boldsymbol{\zeta})'_\xi, \quad (2\ A)$$
$$A_{31} = \mathbf{w}' + (\mathbf{W}\boldsymbol{\zeta})'_\xi, \quad A_{23} = \boldsymbol{\Lambda} + \mathbf{R}\boldsymbol{\zeta}.$$

Here notation $\mathbf{R}\boldsymbol{\zeta}$ stands for $(n-1) \times 2$ matrix with the entries

$$\{\mathbf{R}\boldsymbol{\zeta}\}_{ip} = \{\mathbf{R}_i\}_{pq}\zeta_q, \quad 1 \leq i \leq n-1, \quad p = 1, 2. \quad (3\ A)$$

Substituting (1 A) into the equation $\boldsymbol{\vartheta}'^\top \mathbf{J}_{2n} \boldsymbol{\vartheta}' = \mathbf{J}_{2n}$ we obtain nine matrix equations. Four of those are nontrivial:

$$A_{11}^\top A_{22} = \mathbf{I}, \quad A_{33}^\top \mathbf{J} A_{33} = \mathbf{J}, \quad (4\ A)$$
$$A_{11}^\top A_{23} + A_{31}^\top \mathbf{J} A_{33} = 0, \quad (5\ A)$$
$$A_{11}^\top A_{21} - A_{21}^\top A_{11} + A_{31}^\top \mathbf{J} A_{31} = 0. \quad (6\ A)$$

The remaining five equations are either trivial or can be obtained from (4 A)-(6 A) by transposition. Equations (4 A) along with (2 A) give

$$\mathbf{V} = (\mathbf{I}_{n-1} + \mathbf{u}')^{-\top},$$

$$\mathbf{W}^\top \mathbf{J} \mathbf{W} = \mathbf{J} \text{ or equivalently } \det \mathbf{W} = 1. \tag{7 A}$$

In view of (2 A) we can rewrite equation (5 A) in the form

$$\boldsymbol{\Lambda} + \mathbf{R}\,\boldsymbol{\zeta} = -\boldsymbol{V}\left(\mathbf{w}'_\xi + (\mathbf{W}\boldsymbol{\zeta})'_\xi\right)^\top \mathbf{J}\mathbf{W} \quad \text{for all } \boldsymbol{\zeta} \in \mathbb{R}^2.$$

It follows that

$$\boldsymbol{\Lambda} = -\boldsymbol{V}\left(\mathbf{w}'_\xi\right)^\top \mathbf{J}\mathbf{W}, \tag{8 A}$$

and

$$\mathbf{R}\,\boldsymbol{\zeta} = -\boldsymbol{V}\,(\mathbf{W}\boldsymbol{\zeta})'_\xi{}^\top \mathbf{J}\mathbf{W} \quad \text{for all } \boldsymbol{\zeta} \in \mathbb{R}^2.$$

Recalling the identities

$$\{\mathbf{R}\,\boldsymbol{\zeta}\}_{ij} = R_{i,jq}\,\zeta_q, \quad \{(\mathbf{W}\boldsymbol{\zeta})'_\xi{}^\top\}_{kp} = \frac{\partial}{\partial \xi_k}(W_{pq}\,\zeta_q)$$

we obtain

$$R_{i,qj}\zeta_q = -V_{ik}\,\frac{\partial}{\partial \xi_k}(W_{pq}\zeta_q)\{\mathbf{J}\mathbf{W}\}_{pj} \quad \text{for all } \boldsymbol{\zeta} \in \mathbb{R}^2.$$

Since the matrix $\mathbf{R}_i$ is symmetric, we obtain

$$\mathbf{R}_i = -V_{ik}\,\frac{\partial}{\partial \xi_k}(\mathbf{W}^\top)\,\mathbf{J}\,\mathbf{W}. \tag{9 A}$$

Let us turn to equation (6 A). It is equivalent to the system of four matrix equations

$$\frac{1}{2}\mathbf{V}^{-1}(\boldsymbol{\zeta}^\top \mathbf{R}\boldsymbol{\zeta})'_\xi - \frac{1}{2}(\boldsymbol{\zeta}^\top \mathbf{R}\boldsymbol{\zeta})'_\xi{}^\top \mathbf{V}^{-\top} + (\mathbf{W}\boldsymbol{\zeta})'_\xi{}^\top \mathbf{J}(\mathbf{W}\boldsymbol{\zeta})'_\xi = 0, \tag{10 A}$$

$$\mathbf{V}^{-1}(\boldsymbol{\Lambda}\boldsymbol{\zeta})'_\xi - (\boldsymbol{\Lambda}\boldsymbol{\zeta})'_\xi{}^\top \mathbf{V}^{-\top} + (\mathbf{w})'_\xi{}^\top \mathbf{J}(\mathbf{W}\boldsymbol{\zeta})'_\xi + (\mathbf{W}\boldsymbol{\zeta})'_\xi{}^\top \mathbf{J}(\mathbf{w})'_\xi = 0 \tag{11 A}$$

$$\mathbf{V}^{-1}(\mathbf{V}\boldsymbol{\eta})'_\xi - (\mathbf{V}\boldsymbol{\eta})'_\xi{}^\top \mathbf{V}^{-\top} = 0, \tag{12 A}$$

$$\mathbf{V}^{-1}(\mathbf{v})'_\xi - (\mathbf{v})'_\xi{}^\top \mathbf{V}^{-\top} + (\mathbf{w})'_\xi{}^\top \mathbf{J}(\mathbf{w})'_\xi = 0, \tag{13 A}$$

which hold true for all $\boldsymbol{\zeta} \in \mathbb{R}^2$ and for all $\boldsymbol{\eta} \in \mathbb{R}^{n-1}$. Let us proof that equations (10 A)- (12 A) are consequence of (7 A)-(9 A). We start with equation (10 A). It follows from (7 A) that

$$\mathbf{x} = \boldsymbol{\xi} + \mathbf{u}(\boldsymbol{\xi}), \quad \mathbf{V} = (\mathbf{I} + \mathbf{u}')^{-\top} \equiv (\mathbf{x}'_\xi)^{-\top} = (\boldsymbol{\xi}'_x)^\top,$$

and hence

$$\frac{\partial}{\partial x_i} = V_{ik} \frac{\partial}{\partial \xi_k}, \quad \nabla_x = \mathbf{V} \nabla_\xi. \tag{14 A}$$

Multiplying both sides of equation(10 A) by $\mathbf{V}^\top$ from the right and by $\mathbf{V}$ from the left we can rewrite this equation in the equivalent form

$$\frac{1}{2}(\boldsymbol{\zeta}^\top \mathbf{R} \boldsymbol{\zeta})'_x - \frac{1}{2}(\boldsymbol{\zeta}^\top \mathbf{R} \boldsymbol{\zeta})'^{\top}_x + \left(\mathbf{W} \boldsymbol{\zeta}\right)'^{\top}_x \mathbf{J} \left(\mathbf{W} \boldsymbol{\zeta}\right)'_x = 0 \tag{15 A}$$

In view of (9 A), we have

$$\boldsymbol{\zeta}^\top \mathbf{R}_i \boldsymbol{\zeta} = -\boldsymbol{\zeta}^\top V_{ik} \frac{\partial}{\partial \xi_k}(\mathbf{W}^\top) \mathbf{J} \mathbf{W} \boldsymbol{\zeta} = -\boldsymbol{\zeta}^\top \frac{\partial}{\partial x_i}(\mathbf{W}^\top) \mathbf{J} \mathbf{W} \boldsymbol{\zeta},$$

which implies

$$\{(\boldsymbol{\zeta}^\top \mathbf{R}_i \boldsymbol{\zeta})'_x\}_{ij} = -\boldsymbol{\zeta}^\top \frac{\partial}{\partial x_j} \left(\frac{\partial \mathbf{W}^\top}{\partial x_i} \mathbf{J} \mathbf{W}\right) \boldsymbol{\zeta}.$$

Thus we get

$$\frac{1}{2} \left\{ (\boldsymbol{\zeta}^\top \mathbf{R} \boldsymbol{\zeta})'_x - (\boldsymbol{\zeta}^\top \mathbf{R} \boldsymbol{\zeta})'^{\top}_x \right\}_{ij} = -\frac{1}{2} \boldsymbol{\zeta}^\top \frac{\partial}{\partial x_j} \left(\frac{\partial \mathbf{W}^\top}{\partial x_i} \mathbf{J} \mathbf{W}\right) \boldsymbol{\zeta} +$$
$$\frac{1}{2} \boldsymbol{\zeta}^\top \frac{\partial}{\partial x_i} \left(\frac{\partial \mathbf{W}^\top}{\partial x_j} \mathbf{J} \mathbf{W}\right) \boldsymbol{\zeta} = \frac{1}{2} \boldsymbol{\zeta}^\top \left(\frac{\partial \mathbf{W}^\top}{\partial x_j} \mathbf{J} \frac{\partial \mathbf{W}}{\partial x_i} - \frac{\partial \mathbf{W}^\top}{\partial x_i} \mathbf{J} \frac{\partial \mathbf{W}}{\partial x_j}\right) \boldsymbol{\zeta}. \tag{16 A}$$

On the other hand, we have

$$\{\left(\mathbf{W} \boldsymbol{\zeta}\right)'^{\top}_x \mathbf{J} \left(\mathbf{W} \boldsymbol{\zeta}\right)'_x\}_{ij} = \boldsymbol{\zeta}^\top \left(\frac{\partial \mathbf{W}^\top}{\partial x_i} \mathbf{J} \frac{\partial \mathbf{W}}{\partial x_j}\right) \boldsymbol{\zeta}.$$

Noting that $\boldsymbol{\zeta}^\top \mathbf{A} \boldsymbol{\zeta} = \boldsymbol{\zeta}^\top \mathbf{A}^\top \boldsymbol{\zeta}$ for every matrix $\mathbf{A}$ and setting

$$\mathbf{A} = \frac{\partial \mathbf{W}^\top}{\partial x_i} \mathbf{J} \frac{\partial \mathbf{W}}{\partial x_j}, \quad \mathbf{A}^\top = -\frac{\partial \mathbf{W}^\top}{\partial x_j} \mathbf{J} \frac{\partial \mathbf{W}}{\partial x_i},$$

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we arrive at the identity

$$\{(\mathbf{W}\boldsymbol{\zeta})'_x{}^\top \mathbf{J}(\mathbf{W}\boldsymbol{\zeta})'_x\}_{ij} = \frac{1}{2}\boldsymbol{\zeta}^\top \Big(\frac{\partial \mathbf{W}^\top}{\partial x_i}\mathbf{J}\frac{\partial \mathbf{W}}{\partial x_j} - \frac{\partial \mathbf{W}^\top}{\partial x_j}\mathbf{J}\frac{\partial \mathbf{W}}{\partial x_i}\Big)\boldsymbol{\zeta}. \qquad (17\ A)$$

Combining (16 A) and (17 A) gives (15 A). Hence (7 A) and (9 A) imply (10 A). Let us turn to equation (11 A). Arguing as before we can rewrite it in the equivalent form

$$(\boldsymbol{\Lambda}\boldsymbol{\zeta})'_x - (\boldsymbol{\Lambda}\boldsymbol{\zeta})'_x{}^\top + (\mathbf{w})'_x{}^\top \mathbf{J}(\mathbf{W}\boldsymbol{\zeta})'_x + (\mathbf{W}\boldsymbol{\zeta})'_x{}^\top \mathbf{J}(\mathbf{w})'_x = 0 \qquad (18\ A)$$

Next, equality (8 A) yields

$$\boldsymbol{\Lambda} = -\mathbf{V}\nabla_\xi \mathbf{w}\mathbf{J}\mathbf{W} = -\nabla_x \mathbf{w}\mathbf{J}\mathbf{W}$$

which leads to

$$\{(\boldsymbol{\Lambda}\boldsymbol{\zeta})'_x\}_{ij} = -\frac{\partial}{\partial x_j}\Big(\frac{\partial \mathbf{w}^\top}{\partial x_i}(\mathbf{J}\,\mathbf{W})\,\boldsymbol{\zeta}\Big),$$

and hence

$$\{(\boldsymbol{\Lambda}\boldsymbol{\zeta})'_x - (\boldsymbol{\Lambda}\boldsymbol{\zeta})'_x{}^\top\}_{ij} = \frac{\partial \mathbf{w}^\top}{\partial x_j}\,\mathbf{J}\,\frac{\partial}{\partial x_i}\Big(\mathbf{W}\,\boldsymbol{\zeta}\Big) - \frac{\partial \mathbf{w}^\top}{\partial x_i}\,\mathbf{J}\,\frac{\partial}{\partial x_j}\Big(\mathbf{W}\,\boldsymbol{\zeta}\Big).$$

This relations can be rewritten in the matrix form

$$(\boldsymbol{\Lambda}\boldsymbol{\zeta})'_x - (\boldsymbol{\Lambda}\boldsymbol{\zeta})'_x{}^\top = -(\mathbf{w}'_x)^\top \mathbf{J}(\mathbf{W}\boldsymbol{\zeta})'_x + \big((\mathbf{w}'_x)^\top \mathbf{J}(\mathbf{W}\boldsymbol{\zeta})'_x\big)^\top. \qquad (19\ A)$$

Notice that

$$\big((\mathbf{w}'_x)^\top J(\mathbf{W}\boldsymbol{\zeta})'_x\big)^\top = (\mathbf{W}\boldsymbol{\zeta})'_x{}^\top \mathbf{J}^\top w'_x = -(\mathbf{W}\boldsymbol{\zeta})'_x{}^\top \mathbf{J}^\top w'_x$$

Combining this result with (19 A) we arrive at equation (18 A). Since the latter is equivalent to (11 A), we obtain that (11 A) follows from (7 A)-(8 A). Now consider the equation (12 A). In view of identities (14 A) it can be rewritten in the equivalent form

$$(\mathbf{V}\boldsymbol{\eta})'_\xi \mathbf{V}^\top - \mathbf{V}(\mathbf{V}\boldsymbol{\eta})'_\xi{}^\top \equiv (\mathbf{V}\boldsymbol{\eta})'_x - (\mathbf{V}\boldsymbol{\eta})'_x{}^\top = 0. \qquad (20\ A)$$

On the other hand, formula (7 A) yields

$$\{\mathbf{V}\boldsymbol{\eta}\}_i = \frac{\partial \boldsymbol{\xi}^\top}{\partial x_i}\cdot \boldsymbol{\eta}.$$

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Hence

$$\{(\mathbf{V}\boldsymbol{\eta})'_x\}_{ij} = \frac{\partial^2 \boldsymbol{\xi}^\top}{\partial x_i \partial x_j} \cdot \boldsymbol{\eta}.$$

Hence the matrix $(\mathbf{V}\boldsymbol{\eta})'_x$ is symmetric, and (20 A) is a straightforward consequence of (7 A). Thus we show that equations (10 A)-(12 A) directly follows from (7 A)-(9 A). Let us consider equation (13 A). We prove that it is not trivial and leads to the desired representation $\mathbf{v}$. It follows from (7 A) that equation (13 A) can be written in the form

$$\frac{\partial}{\partial \xi_i}(\xi_k + u_k)\frac{\partial v_k}{\partial \xi_j} - \frac{\partial}{\partial \xi_j}(\xi_k + u_k)\frac{\partial v_k}{\partial \xi_i} + \frac{\partial w_1}{\partial \xi_i}\frac{\partial w_2}{\partial \xi_j} - \frac{\partial w_1}{\partial \xi_j}\frac{\partial w_2}{\partial \xi_i} = 0, \quad 1 \le i, j \le n-1,$$

which is equivalent to

$$d(\xi_k + u_k) \wedge dv_k + dw_1 \wedge w_2 = 0.$$

Next, multiplying (13 A) by $\mathbf{V}$ from the left and by $\mathbf{V}^\top$ from the right we obtain

$$\mathbf{v}'_\xi \mathbf{V}^\top - \mathbf{V}\mathbf{v}'_\xi{}^\top + \mathbf{V}(\mathbf{w}'_\xi)^\top \mathbf{J}\mathbf{w}'_\xi \mathbf{V}^\top \equiv \mathbf{v}_x - \mathbf{v}'_x{}^\top + (\mathbf{w}'_x)^\top \mathbf{J}\mathbf{w}'_x = 0.$$

This means that

$$\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} + \frac{\partial w_1}{\partial x_i}\frac{\partial w_2}{\partial x_j} - \frac{\partial w_1}{\partial x_j}\frac{\partial w_2}{\partial x_i} =$$
$$\frac{\partial}{\partial x_j}\left(v_i + w_2\frac{\partial w_1}{\partial x_i}\right) - \frac{\partial}{\partial x_i}\left(v_j + w_2\frac{\partial w_1}{\partial x_j}\right) = 0$$

Since $\mathbf{v}$ and $\mathbf{w}$ are analytic an 2π-periodic, it follows that there exist analytic 2π-periodic function φ_0 with zero mean value and a constant $\boldsymbol{\beta} \in \mathbb{R}^{n-1}$ such that

$$\beta_i + \frac{\partial \varphi_0}{\partial x_i} = v_i + w_2\frac{\partial w_1}{\partial x_i},$$

this leads to the desired representation for $\mathbf{v}$

$$\mathbf{v} = \beta + \nabla_x \varphi_0 - w_2 \nabla_x w_2 = \beta + \mathbf{V}\left(\nabla_\xi \varphi_0 - w_2 \nabla_\xi w_2\right). \qquad (21\ A)$$

Formulae (7 A)-(9 A) and (21 A) give general solution to equation $\boldsymbol{\vartheta}'^\top \mathbf{J}_{2n}\boldsymbol{\vartheta}' = \mathbf{J}_{2n}$ and completely determine the totality of canonical mappings $\boldsymbol{\vartheta}$. $\square$

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B Proof of Theorem 3.2

B.1 Proof of (i).

Choose an arbitrary analytic

$$\boldsymbol{\varphi} = (\beta, \varphi_0, \mathbf{u}, \mathbf{w}, W_{11}, W_{12}, W_{21}) \tag{1B}$$

and consider the vector field

$$\boldsymbol{\Theta}(\boldsymbol{\varphi}) = \big(\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{V}, \boldsymbol{\Lambda}, \mathbf{W}, \mathbf{R}_i\big), \tag{2B}$$

defined by (3.8). Next, choose an arbitrary

$$\boldsymbol{\Upsilon} = (\boldsymbol{\nu}, \psi_0, \boldsymbol{\chi}, \boldsymbol{\lambda}, \Gamma_{11}, \Gamma_{12}, \Gamma_{21}) \in X_{\sigma, d-1}(r), \tag{3B}$$

and set $\boldsymbol{\mu} = \boldsymbol{\nu} + \nabla\psi_0$, $\Gamma_{22} = -\Gamma_{11}$. Our task is to find the vector field

$$\delta\boldsymbol{\varphi} = (\delta\beta, \delta\varphi_0, \delta\mathbf{u}, \delta\mathbf{w}, \delta W_{11}, \delta W_{12}, \delta W_{21}) \tag{4B}$$

such that the corresponding mapping

$$\delta\boldsymbol{\Theta} = \big(\delta\mathbf{u}, \delta\mathbf{v}, \delta\mathbf{w}, \delta\mathbf{V}, \delta\mathbf{W}, \delta\boldsymbol{\Lambda}, \delta\mathbf{R}_i\big). \tag{5B}$$

given by (3.12) and (3.13), satisfies the equations

$$\delta\mathbf{u}(\boldsymbol{\xi}) = \chi_i(\boldsymbol{\xi})\frac{\partial}{\partial\xi_i}\big(\boldsymbol{\xi} + \mathbf{u}(\boldsymbol{\xi})\big) \Leftrightarrow \boldsymbol{\chi} = \mathbf{V}^\top\delta\mathbf{u}, \tag{6Ba}$$

$$\delta\mathbf{v} = \chi_i\frac{\partial}{\partial\xi_i}\mathbf{v} + \mathbf{V}\boldsymbol{\mu} + \boldsymbol{\Lambda}\boldsymbol{\lambda}, \tag{6Bb}$$

$$\delta\mathbf{w} = \chi_i\frac{\partial}{\partial\xi_i}\mathbf{w} + \mathbf{W}\boldsymbol{\lambda}, \tag{6Bc}$$

$$\delta\mathbf{W} = \chi_i\frac{\partial}{\partial\xi_i}\mathbf{W} + \mathbf{W}\boldsymbol{\Gamma}, \tag{6Bd}$$

and

$$\delta\mathbf{V} = \chi_i\frac{\partial}{\partial\xi_i}\mathbf{V} - \mathbf{V}\nabla_\xi\boldsymbol{\chi}, \tag{7Ba}$$

$$\delta\boldsymbol{\Lambda} = \chi_i\frac{\partial}{\partial\xi_i}\boldsymbol{\Lambda} + \mathbf{V}\nabla_\xi(\mathbf{J}\boldsymbol{\lambda}) + \boldsymbol{\lambda}^\top\mathbf{R} - \boldsymbol{\Lambda}\boldsymbol{\Gamma}, \tag{7Bb}$$

$$\delta\mathbf{R}_i = \chi_p\frac{\partial}{\partial\xi_p}\mathbf{R}_i + \mathbf{R}_i\boldsymbol{\Gamma} + (\mathbf{R}_i\boldsymbol{\Gamma})^\top + V_{ik}\frac{\partial}{\partial\xi_k}(\mathbf{J}\boldsymbol{\Gamma}), \tag{7Bc}$$

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where $\boldsymbol{\lambda}^\top \mathbf{R}$ is $(n-1) \times 2$ matrix with the entries $(\boldsymbol{\lambda}^\top \mathbf{R})_{ij} = R_{i,jp}\, \lambda_p$. Recall that by definition of $\boldsymbol{\Upsilon}$,

$$\boldsymbol{\mu} = \boldsymbol{\nu} + \nabla \psi_0 \tag{7Bd}$$

Notice that relation (6B)-(7B) coincide with desired relations (3.19a)- (3.19g) in Theorem 3.2. In order to obtain the full system of relations (3.19) we have supplement (6B)-(7B) with the expression (3.19h) for $\delta\boldsymbol{\beta}$ and $\delta\varphi_0$:

$$\delta\boldsymbol{\beta} = \boldsymbol{\nu}, \quad \delta\varphi_0 = \psi_0 + w_2 \delta w_1 + \chi_i \frac{\partial}{\partial \xi_i} \varphi_0 - w_2 \chi_i \frac{\partial}{\partial \xi_i} w_1 - \boldsymbol{\nu} \cdot \mathbf{u}, \tag{8B}$$

where $\boldsymbol{\nu}$ and ψ_0 are the components of the given vector $\boldsymbol{\Upsilon}$. The system of equalities (6B)-(8B) is equivalent to (3.19). It is important to note that the left hand sides of these equalities are the components of vector fields $\delta\boldsymbol{\varphi}$ and $\delta\boldsymbol{\Theta}$. They are not independent and should satisfy relations (3.13) which constitutes the connection between $\delta\boldsymbol{\Theta}$ $\delta\boldsymbol{\varphi}$. Hence, relations (3.13) give the extra five equations

$$\delta\mathbf{V} = -\mathbf{V}\,\nabla_\xi \delta\mathbf{u}\,\mathbf{V}, \tag{9Ba}$$

$$\delta W_{22} = \frac{1}{W_{11}}\big(W_{12}\delta W_{21} + W_{21}\delta W_{12} - W_{22}\delta W_{11}\big), \tag{9Bb}$$

$$\delta\mathbf{v} = \delta\beta + \mathbf{V}(\nabla\delta\varphi_0 - w_2\nabla\delta w_1 - \delta w_2 \nabla w_1) + \delta\mathbf{V}\,(\nabla\varphi_0 - w_2\nabla w_1), \tag{9Bc}$$

$$\delta\boldsymbol{\Lambda} = -\delta\mathbf{V}\,\nabla\mathbf{w}\,\mathbf{J}\mathbf{W} - \mathbf{V}\,\nabla(\delta\mathbf{w})\mathbf{J}\mathbf{W} - \mathbf{W}\,\nabla\mathbf{w}\,\mathbf{J}\,\delta\mathbf{W}, \tag{9Bd}$$

$$\delta\mathbf{R}_i = -\delta V_{ik}\frac{\partial}{\partial \xi_k}(\mathbf{W}^\top)\mathbf{J}\mathbf{W} - V_{ik}\frac{\partial}{\partial \xi_k}(\delta\mathbf{W}^\top)\mathbf{J}\mathbf{W} - V_{ik}\frac{\partial}{\partial \xi_k}(\mathbf{W}^\top)\mathbf{J}\,\delta\mathbf{W}. \tag{9Be}$$

The obtained extended system (6B)-(9B) is overdetermined and we have to prove that it satisfies compatibility conditions. The assertion (**i**) of Theorem 3.2 will be proved if we prove that system (9B) is a consequence of equations (6B)- (8B). The proof falls into five steps.

Step 1. Let us show that equalities (6B) and (7B) yield (9Ba). We begin with the observation that equality (6Ba) along with relation (3.4a) in Theorem 3.1 implies $\boldsymbol{\chi} = \mathbf{V}^\top \delta\mathbf{u}$. From this we obtain

$$\chi_k \frac{\partial}{\partial \xi_k}\mathbf{V} = V_{nk}\,\delta u_n \frac{\partial}{\partial \xi_k}\mathbf{V} \tag{10B}$$

Next we have

$$\{\mathbf{V}\nabla\boldsymbol{\chi}\}_{ij} = V_{ik}\frac{\partial}{\partial\xi_k}\big(V_{nj}\,\delta u_n\big) = V_{ik}\frac{\partial V_{nj}}{\partial\xi_k}\,\delta u_n + V_{ik}V_{nj}\frac{\partial\,\delta u_n}{\partial\xi_k}$$

Combining this result with (10B) we arrive at

$$\left\{\chi_k\frac{\partial\mathbf{V}}{\partial\xi_k} - \mathbf{V}\nabla\boldsymbol{\chi}\right\}_{ij} = \left(\frac{V_{nk}\partial V_{ij}}{\partial\xi_k} - V_{ik}\frac{\partial V_{nj}}{\partial\xi_k}\right) - V_{ik}V_{nj}\frac{\partial\delta u_n}{\partial\xi_k}. \qquad (11B)$$

Next notice that $V_{nk}\partial/\partial\xi_k = \partial/\partial x_n$ and $V_{ij} = \partial\xi_j/\partial x_i$. It follows that

$$V_{nk}\frac{\partial V_{ij}}{\partial\xi_k} - V_{ik}\frac{\partial V_{nj}}{\partial\xi_k} = \frac{\partial}{\partial x_n}\Big(\frac{\partial\xi_j}{\partial x_i}\Big) - \frac{\partial}{\partial x_i}\Big(\frac{\partial\xi_j}{\partial x_n}\Big) = 0$$

Substituting this equality into (11B) we arrive at the equality

$$\chi_k\frac{\partial\mathbf{V}}{\partial\xi_k} - \mathbf{V}\nabla\boldsymbol{\xi} = -\mathbf{V}\,\nabla\delta\mathbf{u}\,\mathbf{V},$$

which along with (7Ba) implies (9Ba).

Step 2. Let us prove that (6B) and (7B) imply (9Bb). Notice that (9Bb) is equivalent to the equality

$$\delta\mathbf{W}^\top\mathbf{J}\mathbf{W} + \mathbf{W}^\top\mathbf{J}\delta\mathbf{W} = 0. \qquad (12B)$$

It follows from (6Bd) that

$$\delta\mathbf{W}^\top\mathbf{J}\mathbf{W} + \mathbf{W}^\top\mathbf{J}\delta\mathbf{W} = \chi_i\Big(\frac{\partial}{\partial\xi_i}\mathbf{W}^\top\mathbf{J}\mathbf{W} + \mathbf{W}^\top\mathbf{J}\frac{\partial}{\partial\xi_i}\mathbf{W}\Big) + \\ (\mathbf{W}\boldsymbol{\Gamma})^\top\mathbf{J}\mathbf{W} + \mathbf{W}^\top\mathbf{J}\mathbf{W}\boldsymbol{\Gamma}. \qquad (13B)$$

Next, relation (3.4b) in Theorem 3.1 yields $\mathbf{W}^\top\mathbf{J}\mathbf{W} = \mathbf{J}$. It follows that

$$\frac{\partial}{\partial\xi_i}\mathbf{W}^\top\mathbf{J}\mathbf{W} + \mathbf{W}^\top\mathbf{J}\frac{\partial}{\partial\xi_i}\mathbf{W} = 0$$

and

$$(\mathbf{W}\boldsymbol{\Gamma})^\top\mathbf{J}\mathbf{W} + \mathbf{W}^\top\mathbf{J}\mathbf{W}\boldsymbol{\Gamma} = \boldsymbol{\Gamma}^\top\mathbf{J} + \mathbf{J}\boldsymbol{\Gamma} = \mathbf{J}\boldsymbol{\Gamma} - (\mathbf{J}\boldsymbol{\Gamma})^\top = 0$$

since the matrix $\mathbf{J}\boldsymbol{\Gamma}$ is symmetric and $\mathbf{J}^\top = -\mathbf{J}$. Substituting these relations into (13B) we obtain (12B) and the assertion follows.

Step 3. Let us prove that equalities (6B)-(8B) imply (9Bc). It is necessary to prove that the right hand side of (6Bb) coincide with the right hand side of (9Bc). Let us calculate step by step all terms in the right hand side of (6Bb). Relation (6Ba) and the identity $V_{ik}\partial/\partial\xi_k = \partial/\partial x_i$ imply

$$\chi_i \frac{\partial}{\partial\xi_i}\mathbf{v} = \delta u_i \frac{\partial}{\partial x_i}\mathbf{v}, \tag{14B}$$

On the other hand, relation (3.4f) in Theorem 3.1 yields

$$\mathbf{v} = \boldsymbol{\beta} + \nabla_x\varphi_0 - w_2\nabla_x w_1.$$

Combining these results we arrive at

$$\chi_i \frac{\partial}{\partial\xi_i}\mathbf{v} = \delta u_i \frac{\partial}{\partial x_i}(\nabla_x\varphi_0 - w_2\nabla_x w_1). \tag{15B}$$

Next, relation (3.4c) in Theorem 3.1 implies

$$\boldsymbol{\Lambda} = -\mathbf{V}\,\nabla_\xi\mathbf{w}\,\mathbf{J}\,\mathbf{W} = -\nabla_x\mathbf{w}\,\mathbf{J}\mathbf{W}.$$

In its turn, it follows from (6Bc) that

$$\boldsymbol{\lambda} = \mathbf{W}^{-1}\delta\mathbf{w} - \chi_i\mathbf{W}^{-1}\frac{\partial}{\partial\xi_i}\mathbf{w} = \mathbf{W}^{-1}\delta\mathbf{w} - \delta u_i\mathbf{W}^{-1}\frac{\partial}{\partial x_i}\mathbf{w}. \tag{16B}$$

Thus we get

$$\boldsymbol{\Lambda}\boldsymbol{\lambda} = \nabla_x\mathbf{w}\,\mathbf{J}\Big(\delta u_i\frac{\partial}{\partial x_i}\mathbf{w}\Big) - \nabla_x\mathbf{w}\,\mathbf{J}\,\delta\mathbf{w} = \nabla_x(w_2\delta w_1) -$$

$$(\delta w_2\nabla_x w_1 + w_2\nabla_x\delta w_1) + \Big(\delta u_i\frac{\partial}{\partial x_i}w_2\Big)\nabla_x w_1 - \Big(\delta u_i\frac{\partial}{\partial x_i}w_1\Big)\nabla_x w_2$$

Combining this result with (15B) we arrive at

$$\chi_i \frac{\partial}{\partial\xi_i}\mathbf{v} + \boldsymbol{\Lambda}\boldsymbol{\lambda} = \nabla_x(w_2\delta w_1) - (\delta w_2\nabla_x w_1 + w_2\nabla_x\delta w_1) +$$
$$\delta u_i\,\nabla_x\Big(\frac{\partial}{\partial x_i}\varphi_0 - w_2\frac{\partial}{\partial x_i}w_1\Big). \tag{17B}$$

Next notice that

$$\delta u_i\,\nabla_x\Big(\frac{\partial}{\partial x_i}\varphi_0 - w_2\frac{\partial}{\partial x_i}w_1\Big) =$$

$$\nabla_x\Big(\delta u_i\big(\frac{\partial}{\partial x_i}\varphi_0 - w_2\frac{\partial}{\partial x_i}w_1\big)\Big) - \nabla_x(\delta\mathbf{u})(\nabla_x\varphi_0 - w_2\nabla_x w_1).$$

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Hence we can rewrite (17B) in the form

$$\chi_i \frac{\partial}{\partial \xi_i} \mathbf{v} + \boldsymbol{\Lambda\lambda} = \nabla_x \Big(w_2 \delta w_1 + \delta u_i \frac{\partial}{\partial x_i} \varphi_0 - \delta u_i w_2 \frac{\partial}{\partial x_i} w_1 \Big) - $$
$$(\delta w_2 \nabla_x w_1 + w_2 \nabla_x \delta w_1) - \nabla_x(\delta \mathbf{u})(\nabla_x \varphi_0 - w_2 \nabla_x w_1). \qquad (18B)$$

On the other hand, the identity $\nabla_x = \mathbf{V} \nabla_\xi$ and relation (9Ba), which we have been proved in Step 1, imply

$$\nabla_x(\delta \mathbf{u})(\nabla_x \varphi_0 - w_2 \nabla_x w_1) = -\delta \mathbf{V}(\nabla_\xi \varphi_0 - w_2 \nabla_\xi w_1)$$

Moreover, as it was mentioned above, we have $\delta u_k \partial / \partial x_k = \chi_k \partial / \partial \xi_k$. It follows from this and (18B) that

$$\chi_i \frac{\partial}{\partial \xi_i} \mathbf{v} + \boldsymbol{\Lambda\lambda} = \mathbf{V} \nabla_\xi \Big(w_2 \delta w_1 + \chi_i \frac{\partial}{\partial \xi_i} \varphi_0 - \chi_i w_2 \frac{\partial}{\partial \xi_i} w_1 \Big) - $$
$$\mathbf{V}(\delta w_2 \nabla_\xi w_1 + w_2 \nabla_\xi \delta w_1) + \delta \mathbf{V}(\nabla_\xi \varphi_0 - w_2 \nabla_\xi w_1). \qquad (19B)$$

Let us consider the term $\mathbf{V}\boldsymbol{\mu}$ in expression (6Bb) for $\mathbf{v}$. It follows from (7Bd) that
$$\mathbf{V}\boldsymbol{\mu} = \mathbf{V}\boldsymbol{\nu} + \mathbf{V}\nabla_\xi \psi_0.$$

Next notice that

$$\mathbf{V}\boldsymbol{\nu} = \mathbf{V}\nabla_\xi(\boldsymbol{\nu}\cdot\boldsymbol{\xi}) = \nabla_x(\boldsymbol{\nu}\cdot\boldsymbol{\xi}) = \nabla_x(\boldsymbol{\nu}\mathbf{x} - \boldsymbol{\nu}\cdot\mathbf{u}) = \boldsymbol{\nu} - \nabla_x(\boldsymbol{\nu}\cdot\mathbf{u}) = \boldsymbol{\nu} - \mathbf{V}\nabla_\xi(\boldsymbol{\nu}\cdot\mathbf{u}).$$

Thus we get
$$\mathbf{V}\boldsymbol{\mu} = \boldsymbol{\nu} + \mathbf{V}\nabla_\xi(\psi_0 - \boldsymbol{\nu}\cdot\mathbf{u}).$$

Combining this result with (19B) and recalling formulae (6Bb) for $\delta \mathbf{v}$ and (8B) for φ_0 we finally obtain

$$\delta \mathbf{v} = \chi_i \frac{\partial}{\partial \xi_i} \mathbf{v} + \boldsymbol{\Lambda\lambda} + \mathbf{V}\boldsymbol{\mu} = \boldsymbol{\nu} + \mathbf{V}\nabla \delta \varphi_0 - $$
$$\mathbf{V}(\delta w_2 \nabla_\xi w_1 + w_2 \nabla_\xi \delta w_1) + \delta \mathbf{V}(\nabla_\xi \varphi_0 - w_2 \nabla_\xi w_1),$$

which obviously yields (9Bc).

Step 4. Let us prove that equalities (6B)-(7B) imply (9Bd). It follows from (7Bb) that

$$\delta\mathbf{\Lambda} = \chi_i\frac{\partial\mathbf{\Lambda}}{\partial\xi_i} + \mathbf{V}\nabla_\xi(\mathbf{J}\boldsymbol{\lambda}) + \boldsymbol{\lambda}^\top\mathbf{R} - \mathbf{\Lambda}\mathbf{\Gamma}. \tag{20B}$$

Arguing as in the proof of (14B) and using the identity $V_{ik}\partial/\partial\xi_k = \partial/\partial x_i$ we obtain

$$\chi_i\frac{\partial\mathbf{\Lambda}}{\partial\xi_i} = \delta u_i\frac{\partial\mathbf{\Lambda}}{\partial x_i}, \quad \mathbf{V}\nabla_\xi(\mathbf{J}\boldsymbol{\lambda}) = \nabla_x(\mathbf{J}\boldsymbol{\lambda}). \tag{21B}$$

Thus we get

$$\delta\mathbf{\Lambda} = \delta u_i\frac{\partial\mathbf{\Lambda}}{\partial x_i} + \nabla_x(\mathbf{J}\boldsymbol{\lambda}) + \boldsymbol{\lambda}^\top\mathbf{R} - \mathbf{\Lambda}\mathbf{\Gamma}. \tag{22B}$$

Next, it follows from the relation (3.4d) in Theorem 3.1 that

$$\mathbf{R}_i = -V_{ik}\frac{\partial\mathbf{W}^\top}{\partial\xi_k}\mathbf{J}\,\mathbf{W} = -\frac{\partial\mathbf{W}^\top}{\partial x_i}\mathbf{J}\,\mathbf{W} \tag{23B}$$

Here $\boldsymbol{\lambda}^\top\mathbf{R}$ is $(n-1)\times 2$ -matrix with the rows $\boldsymbol{\lambda}^\top\mathbf{R}_i$, $1 \le i \le n-1$. It follows from (23B) that

$$\boldsymbol{\lambda}^\top\mathbf{R}_i = -\boldsymbol{\lambda}^\top\frac{\partial\mathbf{W}^\top}{\partial x_i}\mathbf{J}\,\mathbf{W} = -\frac{\partial}{\partial x_i}(\boldsymbol{\lambda}^\top\mathbf{W}^\top)\mathbf{J}\mathbf{W} + \frac{\partial\boldsymbol{\lambda}^\top}{\partial x_i}\mathbf{W}^\top\mathbf{J}\mathbf{W}.$$

Noting that in view of relation (3.4b) in Theorem 3.1, we have $\mathbf{W}^\top\mathbf{J}\mathbf{W} = \mathbf{J} = -\mathbf{J}^\top$, we obtain the expression for the rows $\boldsymbol{\lambda}^\top\mathbf{R}_i$ of the matrix $\boldsymbol{\lambda}^\top\mathbf{R}$

$$\boldsymbol{\lambda}^\top\mathbf{R}_i = -\frac{\partial}{\partial x_i}(\mathbf{W}\boldsymbol{\lambda})^\top\mathbf{J}\mathbf{W} - \frac{\partial}{\partial x_i}(\mathbf{J}\boldsymbol{\lambda})^\top.$$

which leads to

$$\boldsymbol{\lambda}^\top\mathbf{R} = -\nabla_x(\mathbf{W}\boldsymbol{\lambda})\mathbf{J}\mathbf{W} - \nabla_x(\mathbf{J}\boldsymbol{\lambda}).$$

On the other hand, equality (6Bc) yields

$$\boldsymbol{\lambda} = \mathbf{W}^{-1}\Big(\delta\mathbf{w} - \chi_i\frac{\partial\mathbf{w}}{\partial\xi_i}\Big) = \mathbf{W}^{-1}\Big(\delta\mathbf{w} - \delta u_i\frac{\partial\mathbf{w}}{\partial x_i}\Big)$$

Substituting this relation into the latter identity we arrive at

$$\boldsymbol{\lambda}^\top\mathbf{R} = -\nabla_x(\delta\mathbf{w})\mathbf{J}\mathbf{W} + \nabla_x\big(\delta u_i\frac{\partial\mathbf{w}}{\partial x_i}\big)\mathbf{J}\mathbf{W} - \nabla_x(\mathbf{J}\boldsymbol{\lambda}). \tag{24B}$$

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Let us calculate $\mathbf{\Lambda\Gamma}$. Equalities (6Bd), the identity $\mathbf{V}\nabla_\xi = \nabla_x$, and equality (9Ba), which we proved in Step 1, implies

$$\mathbf{\Gamma} = \mathbf{W}^{-1}\delta\mathbf{W} - \chi_i\mathbf{W}^{-1}\frac{\partial\mathbf{W}}{\partial\xi_i} = \mathbf{W}^{-1}\delta\mathbf{W} - \delta u_i\mathbf{W}^{-1}\frac{\partial\mathbf{W}}{\partial x_i}. \qquad (25B)$$

On the other hand, equality (3.4c) yields $\mathbf{\Lambda} = -\nabla_x(\mathbf{w})\mathbf{JW}$. Thus we get

$$\mathbf{\Lambda\Gamma} = -\nabla_x(\mathbf{w})\mathbf{J}\delta\mathbf{W} + \delta u_i(\nabla_x\mathbf{w})\mathbf{J}\frac{\partial\mathbf{W}}{\partial x_i}.$$

Substituting this result along with (24B) into (22B) we arrive at

$$\delta\mathbf{\Lambda} = \delta u_i\Big(\frac{\partial\mathbf{\Lambda}}{\partial x_i} + \nabla_x\mathbf{w}\mathbf{J}\frac{\partial\mathbf{W}}{\partial x_i}\Big) + \nabla_x\Big(\delta u_i\frac{\partial\mathbf{w}}{\partial x_i}\Big)\mathbf{JW}- \qquad (26B)$$
$$\nabla_x(\delta\mathbf{w})\mathbf{JW} - \nabla_x\mathbf{w}\,\mathbf{J}\,(\delta\mathbf{W}).$$

Let us calculate separately the term containing δu_i. We begin with the observation that

$$\nabla_x\Big(\delta u_i\frac{\partial\mathbf{w}}{\partial x_i}\Big) = \delta u_i\frac{\partial}{\partial x_i}\nabla_x\mathbf{w} + \nabla_x(\delta\mathbf{u})\,\nabla_x\mathbf{w}.$$

Thus we get

$$\nabla_x\Big(\delta u_i\frac{\partial\mathbf{w}}{\partial x_i}\Big)\mathbf{JW} = \delta u_i\frac{\partial}{\partial x_i}(\nabla_x\mathbf{w})\mathbf{JW} + \nabla_x(\delta\mathbf{u})\,\nabla_x\mathbf{w}\mathbf{JW}. \qquad (27B)$$

Next, the identity $\mathbf{\Lambda} = -\nabla_x\mathbf{w}\,\mathbf{J}\,\mathbf{W}$ implies

$$\delta u_i\frac{\partial\mathbf{\Lambda}}{\partial x_i} = -\delta u_i\frac{\partial}{\partial x_i}(\nabla_x\mathbf{w})\mathbf{JW} - \delta u_i\nabla_x\mathbf{w}\,\mathbf{J}\frac{\partial\mathbf{W}}{\partial x_i}$$

Combining this result with (27B) we obtain

$$\delta u_i\Big(\frac{\partial}{\partial x_i}\mathbf{\Lambda} + \nabla_x\mathbf{w}\mathbf{J}\frac{\partial\mathbf{W}}{\partial x_i}\Big) + \nabla_x\Big(\delta u_i\frac{\partial\mathbf{w}}{\partial x_i}\Big)\mathbf{JW} = \nabla_x(\delta\mathbf{u})\nabla_x\mathbf{w}\,\mathbf{JW}.$$

Substituting this result in (26B) we obtain

$$\delta\mathbf{\Lambda} = -\nabla_x(\delta\mathbf{w})\mathbf{JW} - \nabla_x\mathbf{w}\,\mathbf{J}\,\delta\mathbf{W} + \nabla_x(\delta\mathbf{u})\nabla_x\mathbf{w}\,\mathbf{JW}. \qquad (28B)$$

Next, relation (9Ba), which was proved in Step 1, and the identity $\mathbf{V}\nabla_\xi = \nabla_x$ imply

$$\nabla_x(\delta\mathbf{u})\nabla_x\mathbf{w}\,\mathbf{JW} = (\mathbf{V}\nabla_\xi\delta\mathbf{u})\nabla_x\mathbf{w}\,\mathbf{JW} = -\delta\mathbf{V}\,\nabla_x\mathbf{w}\,\mathbf{J}\,\mathbf{W},$$
$$\nabla_x(\delta\mathbf{w}) = \mathbf{V}\nabla_\xi(\delta\mathbf{w}), \quad \nabla_x\mathbf{w} = \mathbf{V}\nabla_\xi\mathbf{w}.$$

Substituting these equalities into (28B) gives desired relation (9Bd) and the assertion follows.

Step 5. It remains to prove that equalities (6B)-(7B) imply (9Be). We proved yet that $\delta\mathbf{V}$ satisfies relation (9Ba), i.e.,

$$\delta V_{ik} = -V_{in}\Big(\frac{\partial}{\partial\xi_n}\delta u_p\Big)V_{pk}.$$

It follows from this (9Ba) and the identity $\mathbf{V}\nabla_\xi = \nabla_x$ that

$$\delta V_{ik}\partial/\partial\xi_k = -V_{in}\Big(\frac{\partial}{\partial\xi_n}\delta u_p\Big)V_{pk}\partial/\partial\xi_k = -\Big(\frac{\partial}{\partial x_i}\delta u_p\Big)\partial/\partial x_p.$$

It follows from this that relation (9Be) can be rewritten in the equivalent form

$$\delta\mathbf{R}_i = \Big(\frac{\partial}{\partial x_i}\delta u_k\Big)\frac{\partial}{\partial x_k}\big(\mathbf{W}^\top\big)\mathbf{JW} - \frac{\partial}{\partial x_i}\big(\delta\mathbf{W}^\top\big)\mathbf{JW} - \frac{\partial}{\partial x_i}\big(\mathbf{W}^\top\big)\mathbf{J}\delta\mathbf{W}. \quad (29\mathrm{B})$$

Hence it suffices to prove that relations (6B)-(7B) yields (29B). To this end notice that relation (6Ba) and the identity $V_{ik}\partial/\partial\xi_k = \partial/\partial x_i$ imply

$$\chi_k\frac{\partial}{\partial\xi_k}\mathbf{R}_i = \delta u_k\frac{\partial}{\partial x_k}\mathbf{R}_i, \quad V_{ik}\frac{\partial}{\partial\xi_k}(\mathbf{J\,\Gamma}) = \frac{\partial}{\partial x_i}(\mathbf{J\,\Gamma}).$$

Using these identities we can rewrite relation (7Bc) in the form

$$\delta\mathbf{R}_i = \delta u_k\frac{\partial}{\partial x_k}\mathbf{R}_i + \frac{\partial}{\partial x_i}(\mathbf{J\,\Gamma}) + \mathbf{R}_i\mathbf{\Gamma} + (\mathbf{R}_i\mathbf{\Gamma})^\top. \quad (30\mathrm{B})$$

Let us calculate all terms in the right hand side. It follows from (6Bd) and (6Ba) that

$$\delta\mathbf{W} = \chi_i\frac{\partial}{\partial\xi_i}\mathbf{W} + \mathbf{W\Gamma} = \delta u_k\frac{\partial}{\partial x_k}\mathbf{W} + \mathbf{W\Gamma}$$

or equivalently

$$\mathbf{\Gamma} = \mathbf{W}^{-1}\delta\mathbf{W} - \delta u_k\mathbf{W}^{-1}\frac{\partial}{\partial x_k}\mathbf{W}$$

Since $\mathbf{JW}^{-1} = \mathbf{W}^\top\mathbf{J}$ we conclude from this that

$$\mathbf{J\Gamma} = \mathbf{W}^\top\mathbf{J}\delta\mathbf{W} - \delta u_k\mathbf{W}^\top\frac{\partial}{\partial x_k}\mathbf{W},$$

which leads to the equality

$$\frac{\partial}{\partial x_i}(\mathbf{J}\boldsymbol{\Gamma}) = \frac{\partial \mathbf{W}^\top}{\partial x_i}\mathbf{J}\delta\mathbf{W} + \mathbf{W}^\top\mathbf{J}\frac{\partial \delta\mathbf{W}}{\partial x_i} -$$
$$\delta u_k\Big(\frac{\partial \mathbf{W}^\top}{\partial x_i}\mathbf{J}\frac{\partial \mathbf{W}}{\partial x_k} + \mathbf{W}^\top\mathbf{J}\frac{\partial^2 \mathbf{W}}{\partial x_i\partial x_k}\Big) - \frac{\partial \delta u_k}{\partial x_i}\mathbf{W}^\top\mathbf{J}\frac{\partial \mathbf{W}}{\partial x_k}. \tag{31B}$$

Next, equality (23B) implies

$$\delta u_k\frac{\partial}{\partial x_k}\mathbf{R}_i = -\delta u_k\Big(\frac{\partial^2 \mathbf{W}^\top}{\partial x_i\partial x_k}\,\mathbf{J}\mathbf{W} + \frac{\partial \mathbf{W}^\top}{\partial x_i}\,\mathbf{J}\,\frac{\partial \mathbf{W}}{\partial x_k}\Big) \tag{32B}$$

Notice that relations (23B) and (25B) imply

$$\mathbf{R}_i\boldsymbol{\Gamma} + (\mathbf{R}_i\boldsymbol{\Gamma})^\top = \delta\mathbf{W}^\top\mathbf{J}\frac{\partial \mathbf{W}}{\partial x_i} - \frac{\partial \mathbf{W}^\top}{\partial x_i}\,\mathbf{J}\delta\mathbf{W}^\top +$$
$$\delta u_k\Big(\frac{\partial \mathbf{W}^\top}{\partial x_i}\mathbf{J}\frac{\partial \mathbf{W}}{\partial x_k} - \frac{\partial \mathbf{W}^\top}{\partial x_k}\mathbf{J}\frac{\partial \mathbf{W}}{\partial x_i}\Big). \tag{33B}$$

Substituting (31B)-(34B) into (30B) we obtain

$$\delta\mathbf{R}_i = -\frac{\partial \delta u_k}{\partial x_i}\mathbf{W}^\top\mathbf{J}\frac{\partial \mathbf{W}}{\partial x_k} + \mathbf{W}^\top\mathbf{J}\frac{\partial \delta\mathbf{W}}{\partial x_i} + \delta\mathbf{W}^\top\mathbf{J}\frac{\partial \mathbf{W}}{\partial x_i} - \delta u_k\frac{\partial^2}{\partial x_i\partial x_k}\big(\mathbf{W}^\top\mathbf{J}\mathbf{W}\big).$$

On the other hand, the identity $\mathbf{W}^\top\mathbf{J}\mathbf{W} = \mathbf{J}$ implies

$$\frac{\partial^2}{\partial x_i\partial x_k}\big(\mathbf{W}^\top\mathbf{J}\mathbf{W}\big) = 0, \quad \mathbf{W}^\top\mathbf{J}\frac{\partial \mathbf{W}}{\partial x_k} + \frac{\partial \mathbf{W}^\top}{\partial x_k}\mathbf{J}\mathbf{W} = 0.$$

Thus we get

$$\delta\mathbf{R}_i = \frac{\partial \delta u_k}{\partial x_i}\frac{\partial \mathbf{W}^\top}{\partial x_k}\mathbf{J}\mathbf{W} + \mathbf{W}^\top\mathbf{J}\frac{\partial \delta\mathbf{W}}{\partial x_i} + \delta\mathbf{W}^\top\mathbf{J}\frac{\partial \mathbf{W}}{\partial x_i} \tag{34B}$$

Relation (9Bb), which was proved in Step 2, yields $\delta\mathbf{W}^\top\mathbf{J}\mathbf{W} + \mathbf{W}^\top\mathbf{J}\delta\mathbf{W} = 0$ differentiation both sides of this equality gives

$$\mathbf{W}^\top\mathbf{J}\frac{\partial \delta\mathbf{W}}{\partial x_i} + \delta\mathbf{W}^\top\mathbf{J}\frac{\partial \mathbf{W}}{\partial x_i} = -\frac{\partial}{\partial x_i}\big(\delta\mathbf{W}^\top\big)\mathbf{J}\mathbf{W} - \frac{\partial}{\partial x_i}\big(\mathbf{W}^\top\big)\mathbf{J}\delta\mathbf{W}.$$

Substituting this relation into (34B) we obtain desired identity (29B), and the assertion follows.

B.2 Proof of (ii).

Choose an arbitrary analytic

$$\boldsymbol{\varphi} = (\beta, \varphi_0, \mathbf{u}, \mathbf{w}, W_{11}, W_{12}, W_{21}) \tag{35B}$$

and consider the vector field

$$\Theta(\boldsymbol{\varphi}) = (\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{V}, \boldsymbol{\Lambda}, \mathbf{W}, \mathbf{R}_i), \tag{36B}$$

defined by (3.8). Let $\boldsymbol{\theta}$ be a corresponding canonical mapping defined by (3.1). Next choose an arbitrary vector field

$$\delta\boldsymbol{\varphi} = (\delta\beta, \delta\varphi_0, \delta\mathbf{u}, \delta\mathbf{w}, \delta W_{11}, \delta W_{12}, \delta W_{21}) \in X_{\sigma, d-1}. \tag{37B}$$

and the corresponding vector field

$$\delta\boldsymbol{\Theta} = (\delta\mathbf{u}, \delta\mathbf{v}, \delta\mathbf{w}, \delta\mathbf{V}, \delta\mathbf{W}, \delta\boldsymbol{\Lambda}, \delta\mathbf{R}_i). \tag{38B}$$

Let us consider the associated mapping

$$\delta\boldsymbol{\theta} = \begin{pmatrix} \delta\mathbf{u}(\xi) \\ \delta\mathbf{v}(\boldsymbol{\xi}) + \delta\mathbf{V}(\boldsymbol{\xi})\boldsymbol{\eta} + \delta\boldsymbol{\Lambda}(\boldsymbol{\xi})\boldsymbol{\zeta} + \boldsymbol{\zeta}^\top \delta\mathbf{R}(\boldsymbol{\xi})\boldsymbol{\zeta} \\ \delta\mathbf{w}(\boldsymbol{\xi}) + \delta\mathbf{W}(\boldsymbol{\xi})\boldsymbol{\zeta} \end{pmatrix}, \tag{39B}$$

with the coefficients given by (3.13) , i. e.,

$$\delta\mathbf{V} = -\mathbf{V}\,\delta\nabla_\xi \delta\mathbf{u}\,\mathbf{V}, \quad \delta W_{22} = \frac{1}{W_{11}}\big(W_{12}\delta W_{21} + W_{21}\delta W_{12} - W_{22}\delta W_{11}\big),$$

$$\delta\mathbf{W} = \begin{pmatrix} \delta W_{11} & \delta W_{12} \\ \delta W_{21} & \delta W_{22} \end{pmatrix}, \tag{40B}$$

$$\delta\mathbf{v} = \delta\beta + \mathbf{V}(\nabla\delta\varphi_0 - w_2\nabla\delta w_1 - \delta w_2\nabla w_1) + \delta\mathbf{V}\,(\nabla\varphi_0 - w_2\nabla w_1),$$

$$\delta\boldsymbol{\Lambda} = \delta\mathbf{V}\,\nabla w\,\mathbf{J}\mathbf{W} - \mathbf{V}\,\nabla(\delta\mathbf{w})\mathbf{J}\mathbf{W} - \mathbf{W}\,\nabla\mathbf{w}\,\mathbf{J}\,\delta\mathbf{W},$$

$$\delta\mathbf{R}_i = -\delta V_{ik}\frac{\partial}{\partial\xi_k}(\mathbf{W}^\top)\mathbf{J}\mathbf{W} - V_{ik}\frac{\partial}{\partial\xi_k}(\delta\mathbf{W}^\top)\mathbf{J}\mathbf{W} - V_{ik}\frac{\partial}{\partial\xi_k}(\mathbf{W}^\top)\mathbf{J}\,\delta\mathbf{W}.$$

Our task is to find to find a vector field

$$\boldsymbol{\Upsilon} = (\boldsymbol{\nu}, \psi_0, \boldsymbol{\chi}, \boldsymbol{\lambda}, \Gamma_{11}, \Gamma_{12}, \Gamma_{21}) \in X_{\sigma, d-1}(r).$$

such that the corresponding vector field

$$\mathfrak{Z} := \delta\boldsymbol{\Theta} \equiv D_\varphi\boldsymbol{\Theta}(\boldsymbol{\varphi}_0)[\boldsymbol{\Upsilon}], \tag{41B}$$

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$$3 \equiv \Big(\boldsymbol{\chi}, \, \boldsymbol{\mu}, \, \boldsymbol{\lambda}, \, -\nabla\boldsymbol{\chi}, \, \boldsymbol{\Gamma}, \, \nabla(\mathbf{J}\boldsymbol{\lambda}), \, \partial_{\xi_i}(\mathbf{J}\,\boldsymbol{\Gamma}) \Big), \tag{42B}$$

with

$$\boldsymbol{\mu} = \boldsymbol{\nu} + \nabla\psi_0, \quad \operatorname{Tr}\boldsymbol{\Gamma} = 0. \tag{43B}$$

satisfies the relations (3.20), i.e.,

$$\boldsymbol{\chi} = \mathbf{V}^\top \delta\mathbf{u} \tag{44Ba}$$

$$\boldsymbol{\lambda} = \mathbf{W}^{-1}\delta\mathbf{w} - \chi_i\,\mathbf{W}^{-1}\frac{\partial}{\partial\xi_i}\mathbf{w} \tag{44Bb}$$

$$\boldsymbol{\Gamma} = \mathbf{W}^{-1}\delta\mathbf{W} - \chi_i\,\mathbf{W}^{-1}\frac{\partial}{\partial\xi_i}\mathbf{W}, \tag{44Bc}$$

$$\boldsymbol{\mu} = \mathbf{V}^{-1}\Big(\delta\mathbf{W} + \chi_i\,\frac{\partial}{\partial\xi_i}\mathbf{v} - \boldsymbol{\Lambda}\boldsymbol{\lambda}\Big) \tag{44Bd}$$

$$\nabla\psi_0 = \boldsymbol{\mu} - \boldsymbol{\nu}, \quad \boldsymbol{\nu} = \frac{1}{(2\pi)^{n-1}}\int_{\mathbb{T}^{n-1}}\boldsymbol{\mu}\,d\boldsymbol{\xi} = \delta\boldsymbol{\beta}. \tag{44Be}$$

In order to prove (44B), we consider the associated canonical mapping

$$\boldsymbol{\pi} = \begin{pmatrix} \boldsymbol{\chi}(\boldsymbol{\xi}) \\ \boldsymbol{\mu} - (\nabla_\xi\boldsymbol{\chi})\boldsymbol{\eta} + \nabla_\xi(\mathbf{J}\boldsymbol{\lambda})\boldsymbol{\zeta} + \frac{1}{2}\nabla_\xi(\mathbf{J}\boldsymbol{\Gamma}\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}) \\ \boldsymbol{\lambda} + \boldsymbol{\Gamma}\boldsymbol{\zeta} \end{pmatrix}. \tag{45B}$$

Notice that relations (44B) can be written in the form

$$\delta\boldsymbol{\theta} = \boldsymbol{\theta}'\,\boldsymbol{\pi}. \tag{46B}$$

Here the Jacobi matrix $\boldsymbol{\theta}'$ is defined by (1 A)-(2 A). Since $\boldsymbol{\theta}$ and $\delta\boldsymbol{\theta}$ are given, it suffices to prove that a solution $\boldsymbol{\pi}$ to (46B) admits representation (45B). Since, in view of (3.3), the matrix $\boldsymbol{\theta}'$ is symplectic, a solution of (46B) is given by

$$\boldsymbol{\pi} = -\mathbf{J}_{2n}\boldsymbol{\vartheta}'^\top\mathbf{J}_{2n}\delta\boldsymbol{\vartheta}. \tag{47B}$$

Here the matrix $\mathbf{J}_{2n}$ is defined by (3.3). Substituting (39B) into (47B) we conclude that the mapping $\boldsymbol{\pi}$ admits the representation

$$\boldsymbol{\pi} = \begin{pmatrix} \boldsymbol{\chi}(\boldsymbol{\xi}) \\ \boldsymbol{\mu}(\boldsymbol{\xi}) + \mathbf{A}(\boldsymbol{\xi})\boldsymbol{\eta} + \mathbf{B}(\boldsymbol{\xi})\boldsymbol{\zeta} + \frac{1}{2}\boldsymbol{\zeta}^\top C(\boldsymbol{\xi})\boldsymbol{\zeta} \\ \boldsymbol{\lambda}(\boldsymbol{\xi}) + \boldsymbol{\Gamma}(\boldsymbol{\xi})\boldsymbol{\zeta} \end{pmatrix}. \tag{48B}$$

Here $\boldsymbol{\chi}$, $\boldsymbol{\mu}$, $\boldsymbol{\lambda}$ are 2π- periodic vector fields, $\mathbf{A}$, $\mathbf{B}$ and $\boldsymbol{\Gamma}$ are 2π periodic matrices, and a vector-valued function $\boldsymbol{\zeta}^{\top}C(\boldsymbol{\xi})\boldsymbol{\zeta}$ is in the form

$$\boldsymbol{\zeta}^{\top}C(\boldsymbol{\xi})\boldsymbol{\zeta} = \left(\boldsymbol{\zeta}^{\top}\mathbf{C}_1(\boldsymbol{\xi})\boldsymbol{\zeta}, \ldots, \boldsymbol{\zeta}^{\top}\mathbf{C}_{n-1}(\boldsymbol{\xi})\boldsymbol{\zeta}\right), \tag{49B}$$

where $\mathbf{C}_i$ are 2π-periodic matrices. It follows directly from (48B), (39B), and expression (3.3) for the Jacobi matrix $\boldsymbol{\theta}'$ that

$$\boldsymbol{\chi} = (\mathbf{I}_{n-1} + \mathbf{u}')^{-1}\delta\mathbf{u},$$

$$\boldsymbol{\lambda} = \mathbf{W}^{-1}\delta\mathbf{w} - \chi_i\mathbf{W}^{-1}\frac{\partial}{\partial\xi_i}\mathbf{w},$$

$$\boldsymbol{\mu} = \mathbf{V}^{-1}\left(\delta\mathbf{v} - \chi_i\frac{\partial}{\partial\xi_i}\mathbf{v} - \boldsymbol{\Lambda}\boldsymbol{\lambda}\right), \tag{50B}$$

$$\boldsymbol{\Gamma} = \mathbf{W}^{-1}\delta\mathbf{w} - \chi_i\mathbf{W}^{-1}\frac{\partial}{\partial\xi_i}\mathbf{W}.$$

Hence it remains to prove that

$$\mathbf{A} = -\nabla\boldsymbol{\chi}, \quad \mathbf{B} = \nabla(\mathbf{J}\boldsymbol{\lambda}) \tag{51B}$$

$$\mathbf{C}_i = \frac{\partial}{\partial\xi_i}(\mathbf{J}\boldsymbol{\Gamma}), \quad \mathbf{J}\boldsymbol{\Gamma} + \boldsymbol{\Gamma}^{\top}\mathbf{J} = 0, \tag{52B}$$

$$\boldsymbol{\mu} = \boldsymbol{\nu} + \nabla\psi_0. \tag{53B}$$

To this end notice that $\boldsymbol{\theta}'^{\top}\mathbf{J}_{2n}\boldsymbol{\theta}' = \mathbf{J}_{2n}$, which yields

$$(\delta\boldsymbol{\theta})'^{\top}\mathbf{J}_{2n}\,\boldsymbol{\theta}' + \boldsymbol{\theta}'^{\top}\mathbf{J}_{2n}\,(\delta\boldsymbol{\theta})' = 0. \tag{54B}$$

Now set $\boldsymbol{\sigma} = (\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\zeta}) \in R^{2n}$ and notice that the vector-valued function $\boldsymbol{\pi}$ takes it value in $\mathbb{R}^{2n}$. Denote by π_k the components of $\boldsymbol{\pi}$. Differentiation of the equality $\delta\boldsymbol{\theta} = \boldsymbol{\theta}'\boldsymbol{\pi}$ leads to the identity

$$\delta\boldsymbol{\theta}' = \boldsymbol{\theta}'\boldsymbol{\pi}' + \pi_k\frac{\partial}{\partial\sigma_k}\boldsymbol{\theta}'.$$

Substituting this result in (10B) we arrive at

$$\boldsymbol{\pi}'^{\top}\boldsymbol{\theta}'^{\top}\mathbf{J}_{2n}\boldsymbol{\theta}' + \boldsymbol{\theta}'^{\top}\mathbf{J}_{2n}\boldsymbol{\theta}'\boldsymbol{\pi}' + \pi_k\left\{\boldsymbol{\theta}'^{\top}\mathbf{J}_{2n}\frac{\partial}{\partial\sigma_k}\boldsymbol{\theta}' + \frac{\partial}{\partial\sigma_k}\boldsymbol{\theta}'^{\top}\mathbf{J}_{2n}\boldsymbol{\theta}'\right\}. = 0.$$

Notice that

$$\boldsymbol{\theta'}^\top \mathbf{J}_{2n} \frac{\partial}{\partial \sigma_k} \boldsymbol{\theta'} + \frac{\partial}{\partial \sigma_k} \boldsymbol{\theta'}^\top \mathbf{J}_{2n} \, \boldsymbol{\theta'} = \frac{\partial}{\partial \sigma_k} \big(\boldsymbol{\theta'}^\top \mathbf{J}_{2n} \, \boldsymbol{\theta'}\big) \equiv \frac{\partial}{\partial \sigma_k} \mathbf{J}_{2n} = 0$$

Recalling that $\boldsymbol{\theta'}^\top \mathbf{J}_{2n} \, \boldsymbol{\theta'} = \mathbf{J}_{2n}$ we obtain

$$\boldsymbol{\pi'}^\top \mathbf{J}_{2n} + \mathbf{J}_{2n} \, \boldsymbol{\pi'} = 0. \tag{55B}$$

It follows from (48B) that

$$\boldsymbol{\pi'} = \begin{pmatrix} P_{11}, & 0, & 0 \\ P_{21}, & P_{22}, & P_{23} \\ P_{31}, & 0, & P_{33} \end{pmatrix}, \quad \mathbf{J}_{2n} = \begin{pmatrix} 0 & \mathbf{I}_{n-1} & 0 \\ -\mathbf{I}_{n-1} & 0 & 0 \\ 0 & 0 & \mathbf{J} \end{pmatrix}, \tag{56B}$$

where
$$P_{11} = \boldsymbol{\chi}', \quad P_{21}\big(\boldsymbol{\mu} + \mathbf{A}\boldsymbol{\eta} + \mathbf{B}\boldsymbol{\zeta} + \frac{1}{2}\boldsymbol{\zeta}^\top C \boldsymbol{\zeta}\big)'_\xi$$

$$P_{22} = \mathbf{A} \quad P_{23} = \mathbf{B} + \mathbf{C}\boldsymbol{\zeta},$$

$$P_{31} = \boldsymbol{\lambda}'_\xi + (\boldsymbol{\Gamma}\boldsymbol{\zeta})'_\xi, \quad P_{33} = \boldsymbol{\Gamma},$$

Here $\mathbf{C}\boldsymbol{\lambda}$ is $(n-1)\times 2$ matrix with the entries $(\mathbf{C}\boldsymbol{\lambda})_{ij} = C_{i,jp}\,\lambda_p$. Substituting (56B) into (55B) we obtain four nontrivial matrix equations

$$P_{22} = -P_{11}^\top, \quad P_{23} = (\mathbf{J}P_{31}^\top,$$

$$P_{33}^\top \mathbf{J} + \mathbf{J}P_{33} = 0, \quad P_{21} = P_{21}^\top.$$

The first equation gives $\mathbf{A} = -\nabla\boldsymbol{\chi}$. The second gives two nontrivial relation

$$\mathbf{B} = (\mathbf{J}\boldsymbol{\lambda})'_\xi{}^\top \equiv \nabla(\mathbf{J}\boldsymbol{\lambda}), \quad \mathbf{C}\boldsymbol{\zeta} = \nabla_\xi(\mathbf{J}\boldsymbol{\Gamma}\boldsymbol{\zeta}).$$

The latter equality is equivalent to relations $\mathbf{C}_i = \partial(\mathbf{J}\boldsymbol{\Gamma})/\partial\xi_i$. Finally notice that relation $P_{21} = P_{21}^\top$ gives $\boldsymbol{\mu}'_\xi = (\boldsymbol{\mu}'_\xi)^\top$. In other words, the Jacobi matrix of the vector-field $\boldsymbol{\mu}$ is symmetric. It follows that the vector field $\boldsymbol{\mu}$ is potential. Hence it has the representation $\boldsymbol{\mu} = \boldsymbol{\nu} + \nabla\psi_0$, where $\boldsymbol{\nu}$ is a constant vector and ψ_0 is 2π-periodic function with zero mean. Therefore the matrices $\mathbf{A}$, $\mathbf{B}$, $\boldsymbol{\Gamma}$ and the vector field $\boldsymbol{\mu}$ satisfy conditions (51B)-(53B). This completes the proof of Theorem 3.2.

C   Proof of Lemma 4.1

Equation (4.9b) along with the equality $\mathbf{V}^{-1} = \nabla(\boldsymbol{\xi} + \mathbf{u})$ imply

$$\boldsymbol{\omega}^\top \nabla(\boldsymbol{\xi} + \mathbf{u}) = \nabla_y H_m(\boldsymbol{\xi} + \mathbf{u}, \mathbf{v}, \mathbf{w}),$$

which yields the first relation in (4.11). Next, recalling formula (4.9f) for $\boldsymbol{\Lambda}$
, we obtain

$$\left(\frac{\partial H_m}{\partial \mathbf{y}}(\boldsymbol{\xi} + \mathbf{u}, \mathbf{v}, \mathbf{w})\boldsymbol{\Lambda} \right)_q = -\frac{\partial H_m}{\partial y_i}(\boldsymbol{\xi} + \mathbf{u}, \mathbf{v}, \mathbf{w})V_{ij}\frac{\partial w_k}{\partial \xi_j}J_{kp}W_{pq},$$

We also have

$$\left(\frac{\partial H_m}{\partial \mathbf{z}}(\boldsymbol{\xi} + \mathbf{u}, \mathbf{v}, \mathbf{w})W \right)_q = \frac{\partial H_m}{\partial z_p}(\boldsymbol{\xi} + \mathbf{u}, \mathbf{v}, \mathbf{w})W_{pq}$$

From this and (4.9c) we obtain

$$\left\{ -\frac{\partial H_m}{\partial y_i}(\boldsymbol{\xi} + \mathbf{u}, \mathbf{v}, \mathbf{w})V_{ij}\frac{\partial w_k}{\partial \xi_j}J_{kp} + \frac{\partial H_m}{\partial z_p}(\boldsymbol{\xi} + \mathbf{u}, \mathbf{v}, \mathbf{w}) \right\}W_{pq} = 0.$$

Notice that in view of (4.9b), we have $\partial H_m/\partial y_i V_{ij} = \omega_j$. Thus we get

$$\left\{ -\boldsymbol{\partial}\mathbf{w}^\top \mathbf{J} + \frac{\partial H_m}{\partial \mathbf{z}} \right\}\mathbf{W} = 0.$$

which yields the third inequality in (4.11). In order to derive the second equality in (4.11) we differentiate both sides of (4.9a) with respect to x_i to obtain

$$\frac{\partial H_m}{\partial x_i}(\xi + \mathbf{u}, \mathbf{v}, \mathbf{w}) + \frac{\partial H_m}{\partial y_n}(\xi + \mathbf{u}, \mathbf{v}, \mathbf{w})\frac{\partial v_n}{\partial x_i} + \frac{\partial H_m}{\partial z_p}(\xi + \mathbf{u}, \mathbf{v}, \mathbf{w})\frac{\partial w_p}{\partial x_i} = 0. \quad (1\text{C})$$

Since $\mathbf{V} = (\mathbf{I} + \mathbf{u}')^{-\top}$, we have $\nabla_x = \mathbf{V}\nabla_\xi$. It follows that

$$\frac{\partial}{\partial y_n}H_m(\xi + \mathbf{u}, \mathbf{v}, \mathbf{w})\frac{\partial}{\partial x_n} = \frac{\partial}{\partial y_n}H_m(\xi + \mathbf{u}, \mathbf{v}, \mathbf{w})V_{nj}\frac{\partial}{\partial \xi_j}.$$

From this and (4.9b) we obtain

$$\frac{\partial}{\partial y_n}H_m(\xi + \mathbf{u}, \mathbf{v}, \mathbf{w})\frac{\partial}{\partial x_n} = \omega_i\frac{\partial}{\partial \xi_i} = \boldsymbol{\partial}. \quad (2\text{C})$$

On the other hand, relation (4.9f) implies $\mathbf{v} = \boldsymbol{\beta} + \nabla_x \varphi_0 - w_1 \nabla_x w_2$. Thus we get

$$\frac{\partial v_n}{\partial x_i} = \frac{\partial^2 \varphi_0}{\partial x_n \partial x_i} - w_1 \frac{\partial^2 w_2}{\partial x_n \partial x_i} - \frac{\partial w_1}{\partial x_n} \frac{\partial w_2}{\partial x_i}.$$

Combining this identity with (2C) we arrive at

$$\frac{\partial}{\partial y_n} H_m(\xi + \mathbf{u}, \mathbf{v}, \mathbf{w}) \frac{\partial v_n}{\partial x_i} = \boldsymbol{\partial} \frac{\partial \varphi_0}{\partial x_i}$$

$$- w_2 \boldsymbol{\partial} \frac{\partial w_1}{\partial x_i} - \boldsymbol{\partial} w_1 \frac{\partial w_2}{\partial x_i} =$$

$$\boldsymbol{\partial} v_i - \boldsymbol{\partial} w_1 \frac{\partial w_2}{\partial x_i} + \boldsymbol{\partial} w_2 \frac{\partial w_1}{\partial x_i}$$

From this and the latter relation in (4.11) we obtain

$$\frac{\partial H_m}{\partial y_n}(\xi + \mathbf{u}, \mathbf{v}, \mathbf{w}) \frac{\partial v_n}{\partial x_i} =$$

$$\boldsymbol{\partial} v_i - \frac{\partial H_m}{\partial z_2} \frac{\partial w_2}{\partial x_i} - \frac{\partial H_m}{\partial z_1} \frac{\partial w_1}{\partial x_i}$$

Substituting this equality into (1C) we obtain the second equality in (4.11).
$\square$

D   Proof of Theorem 4.7

Direct calculations lead to the following formulae

$$\delta \Phi_1 = \frac{\partial H_m}{\partial \mathbf{x}}(\boldsymbol{\xi} + \mathbf{u}, \mathbf{v}, \mathbf{w}) \delta \mathbf{u} +$$

$$\frac{\partial H_m}{\partial \mathbf{y}}(\boldsymbol{\xi} + \mathbf{u}, \mathbf{v}, \mathbf{w}) \delta \mathbf{v} + \frac{\partial H_m}{\partial \mathbf{z}}(\boldsymbol{\xi} + \mathbf{u}, \mathbf{v}, \mathbf{w}) \delta \mathbf{w},$$

(1Da)

$$\delta \Phi_2 = \Big\{ \frac{\partial H_m}{\partial \mathbf{y}}(\boldsymbol{\xi} + \mathbf{u}, \mathbf{v}, \mathbf{w}) \delta \boldsymbol{\Lambda} + \frac{\partial H_m}{\partial \mathbf{z}}(\boldsymbol{\xi} + \mathbf{u}, \mathbf{v}, \mathbf{w}) \delta \mathbf{W} \Big\}^{\top} +$$

$$\Big\{ (\delta \mathbf{u})^{\top} \Big(\frac{\partial^2 H_m}{\partial \mathbf{x} \partial \mathbf{y}}(\boldsymbol{\xi} + \mathbf{u}, \mathbf{v}, \mathbf{w}) \boldsymbol{\Lambda} + \frac{\partial^2 H_m}{\partial \mathbf{x} \partial \mathbf{z}}(\boldsymbol{\xi} + \mathbf{u}, \mathbf{v}, \mathbf{w}) \mathbf{W} \Big) \Big\}^{\top} +$$

$$\Big\{ (\delta \mathbf{v})^{\top} \Big(\frac{\partial^2 H_m}{\partial \mathbf{y}^2}(\boldsymbol{\xi} + \mathbf{u}, \mathbf{v}, \mathbf{w}) \boldsymbol{\Lambda} + \frac{\partial^2 H_m}{\partial \mathbf{y} \partial \mathbf{z}}(\boldsymbol{\xi} + \mathbf{u}, \mathbf{v}, \mathbf{w}) \mathbf{W} \Big) \Big\}^{\top} +$$

$$\Big\{ (\delta \mathbf{w})^{\top} \Big(\frac{\partial^2 H_m}{\partial \mathbf{z} \partial \mathbf{y}}(\boldsymbol{\xi} + \mathbf{u}, \mathbf{v}, \mathbf{w}) \boldsymbol{\Lambda} + \frac{\partial^2 H_m}{\partial \mathbf{z}^2}(\boldsymbol{\xi} + \mathbf{u}, \mathbf{v}, \mathbf{w}) \mathbf{W} \Big) \Big\}^{\top},$$

(1Db)

$$\delta\Phi_3 = \{\frac{\partial H_m}{\partial \mathbf{y}}(\boldsymbol{\xi}+\mathbf{u},\mathbf{v},\mathbf{w})\delta\mathbf{V} + (\delta\mathbf{u})^\top\frac{\partial^2 H_m}{\partial\mathbf{x}\partial\mathbf{y}}(\boldsymbol{\xi}+\mathbf{u},\mathbf{v},\mathbf{w})\}^\top +$$
$$\{(\delta\mathbf{v})^\top\frac{\partial^2 H_m}{\partial\mathbf{y}^2}(\boldsymbol{\xi}+\mathbf{u},\mathbf{v},\mathbf{w})\mathbf{V} + (\delta\mathbf{w})^\top\frac{\partial^2 H_m}{\partial\mathbf{z}\partial\mathbf{y}}(\boldsymbol{\xi}+\mathbf{u},\mathbf{v},\mathbf{w})\mathbf{V}\}^\top, \tag{1Dc}$$

$$\delta\Phi_4 = \frac{\partial H_m}{\partial y_i}(\boldsymbol{\xi}+\mathbf{u},\mathbf{v},\mathbf{w})\delta\mathbf{R}_i + (\mathbf{A}+\mathbf{B}+\mathbf{C}) + (\mathbf{A}+\mathbf{B}+\mathbf{C})^\top +$$
$$(\delta\mathbf{u})^\top\mathbf{P}^{(x)} + (\delta\mathbf{v})^\top\mathbf{P}^{(y)} + (\delta\mathbf{w})^\top\mathbf{P}^{(z)}, \tag{1Dd}$$

$$\delta\Phi_5 = \frac{1}{(2\pi)^{n-1}}\int_{\mathbb{T}^{n-1}}\delta w_1\,d\boldsymbol{\xi}, \tag{1De}$$

where

$$\mathbf{A} = \delta\boldsymbol{\Lambda}^\top\frac{\partial^2 H_m}{\partial\mathbf{y}^2}(\boldsymbol{\xi}+\mathbf{u},\mathbf{v},\mathbf{w})\boldsymbol{\Lambda},$$

$$\mathbf{B} = \delta\mathbf{W}^\top\frac{\partial^2 H_m}{\partial\mathbf{z}\partial\mathbf{y}}(\boldsymbol{\xi}+\mathbf{u},\mathbf{v},\mathbf{w})\boldsymbol{\Lambda} + \mathbf{W}^\top\frac{\partial^2 H_m}{\partial\mathbf{z}\partial\mathbf{y}}(\boldsymbol{\xi}+\mathbf{u},\mathbf{v},\mathbf{w})\delta\boldsymbol{\Lambda}, \tag{1Df}$$

$$\mathbf{C} = \delta\mathbf{W}^\top\frac{\partial^2 H_m}{\partial\mathbf{z}^2}(\boldsymbol{\xi}+\mathbf{u},\mathbf{v},\mathbf{w})\mathbf{W},$$

$$\mathbf{P}^{(\tau)} = \boldsymbol{\Lambda}^\top\frac{\partial^3 H_m}{\partial\boldsymbol{\tau}\partial\mathbf{y}^2}(\boldsymbol{\xi}+\mathbf{u},\mathbf{v},\mathbf{w})\boldsymbol{\Lambda} + \boldsymbol{\Lambda}^\top\frac{\partial^3 H_m}{\partial\boldsymbol{\tau}\partial\mathbf{y}\partial\mathbf{z}}(\boldsymbol{\xi}+\mathbf{u},\mathbf{v},\mathbf{w})\mathbf{W} +$$
$$\mathbf{W}^\top\frac{\partial^3 H_m}{\partial\boldsymbol{\tau}\partial\mathbf{z}\partial\mathbf{y}}(\boldsymbol{\xi}+\mathbf{u},\mathbf{v},\mathbf{w})\boldsymbol{\Lambda} + \mathbf{W}^\top\frac{\partial^3 H_m}{\partial\boldsymbol{\tau}\partial\mathbf{z}^2}(\boldsymbol{\xi}+\mathbf{u},\mathbf{v},\mathbf{w})\mathbf{W}, \tag{1Dg}$$

where $\boldsymbol{\tau} = \mathbf{x}, \mathbf{y}, \mathbf{z}$. The variations $\delta\mathbf{V}$, $\delta\mathbf{W}$, $\delta\boldsymbol{\Lambda}$, and $\delta\mathbf{v}$ are defined in terms of the components of $\delta\boldsymbol{\varphi}$ by relations (3.13).

In view of the Second Structure Theorem (3.2) the mapping (4.22) has the inverse given by formulae (3.19). In particular, we have

$$\delta\mathbf{u} = \chi_i\frac{\partial}{\partial\xi_i}(\boldsymbol{\xi}+\mathbf{u}), \quad \delta\mathbf{v} = \mathbf{V}\boldsymbol{\mu} + \boldsymbol{\Lambda}\boldsymbol{\lambda} + \chi_i\frac{\partial}{\partial\xi_i}\mathbf{v},$$
$$\delta\mathbf{w} = \mathbf{W}\boldsymbol{\lambda} + \chi_i\frac{\partial}{\partial\xi_i}\mathbf{w}. \tag{2D}$$

Substituting these equalities in representation (1Da) for $\delta\Phi_1$ we arrive at

$$\delta\Phi_1 = \frac{\partial H_m}{\partial\mathbf{y}}\mathbf{V}\boldsymbol{\mu} + \Big(\frac{\partial H_m}{\partial\mathbf{x}}\frac{\partial}{\partial\xi_i}(\mathrm{Id}+\mathbf{u}) + \frac{\partial H_m}{\partial\mathbf{y}}\frac{\partial\mathbf{v}}{\partial\xi_i} + \frac{\partial H_m}{\partial\mathbf{z}}\frac{\partial\mathbf{w}}{\partial\xi_i}\Big)\chi_i +$$
$$\Big(\frac{\partial H_m}{\partial\mathbf{y}}\boldsymbol{\Lambda} + \frac{\partial H_m}{\partial\mathbf{z}}\mathbf{W}\Big)\boldsymbol{\lambda} \tag{3D}$$

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Here the derivatives of H_m are calculated at the point $(\boldsymbol{\xi}+\mathbf{u}, \mathbf{v}, \mathbf{w})$. It follows from formula (4.9a) for Φ_1 that

$$\frac{\partial H_m}{\partial \mathbf{x}}\frac{\partial}{\partial \xi_i}(\mathrm{Id}+\mathbf{u}) + \frac{\partial H_m}{\partial \mathbf{y}}\frac{\partial \mathbf{v}}{\partial \xi_i} + \frac{\partial H_m}{\partial \mathbf{z}}\frac{\partial \mathbf{w}}{\partial \xi_i} = \frac{\partial \Phi_1}{\partial \xi_i}. \tag{4D}$$

Next, formula (4.9c) for Φ_3 and the equality $\boldsymbol{\mu} = \nabla\psi_0 + \delta\boldsymbol{\beta}$ imply

$$\frac{\partial H_m}{\partial \mathbf{y}}\mathbf{V}\boldsymbol{\mu} = \boldsymbol{\omega}^\top\boldsymbol{\mu} + \Big(\frac{\partial H_m}{\partial \mathbf{y}}\mathbf{V} - \boldsymbol{\omega}^\top\Big)\boldsymbol{\mu} = \boldsymbol{\partial}\psi_0 + \boldsymbol{\omega}^\top\cdot\delta\boldsymbol{\beta} + \Phi_3^\top\boldsymbol{\mu}. \tag{5D}$$

Formula (4.9b) for Φ_2 yields

$$\Big(\frac{\partial H_m}{\partial \mathbf{y}}\boldsymbol{\Lambda} + \frac{\partial H_m}{\partial \mathbf{z}}\mathbf{W}\Big)\lambda = \Phi_2^\top\boldsymbol{\lambda}. \tag{6D}$$

Substituting (4D)-(6D) into (3D) we obtain the desired representation (4.23a) for $\delta\Phi_1$. Let us turn to the representation for $\delta\Phi_2$. Equalities (3.19d), (3.19f) in the second structural theorem imply

$$\delta\boldsymbol{\Lambda} = \mathbf{V}\nabla_\xi(\mathbf{J}\boldsymbol{\lambda}) + \chi_i\frac{\partial}{\partial \xi_i}\boldsymbol{\Lambda} + \boldsymbol{\lambda}^\top\mathbf{R} + \boldsymbol{\Lambda}\boldsymbol{\Gamma},$$

$$\delta\mathbf{W} = \mathbf{W}\boldsymbol{\Gamma} + \chi_i\frac{\partial}{\partial \xi_i}\mathbf{W}. \tag{7D}$$

Substituting these relations along with equalities (2D) into (1Db) we obtain

$$\delta\Phi_2^\top = \frac{\partial H_m}{\partial \mathbf{y}}\mathbf{V}\nabla_\xi(\mathbf{J}\boldsymbol{\lambda}) + \frac{\partial H_m}{\partial \mathbf{y}}(\boldsymbol{\lambda}^\top\mathbf{R})+$$

$$\chi_i\frac{\partial H_m}{\partial \mathbf{y}}\frac{\partial \boldsymbol{\Lambda}}{\partial \xi_i} + \frac{\partial H_m}{\partial \mathbf{y}}\boldsymbol{\Lambda}\boldsymbol{\Gamma} + \chi_i\frac{\partial H_m}{\partial \mathbf{z}}\frac{\partial \mathbf{W}}{\partial \xi_i} + \frac{\partial H_m}{\partial \mathbf{z}}\mathbf{W}\boldsymbol{\Gamma}+$$

$$\chi_i\frac{\partial}{\partial \xi_i}(\boldsymbol{\xi}+\mathbf{u})^\top\Big(\frac{\partial^2 H_m}{\partial \boldsymbol{x}\partial \boldsymbol{y}}\boldsymbol{\Lambda} + \frac{\partial^2 H_m}{\partial \boldsymbol{x}\partial \boldsymbol{z}}\mathbf{W}\Big)+$$

$$\chi_i\frac{\partial \mathbf{v}^\top}{\partial \xi_i}\Big(\frac{\partial^2 H_m}{\partial \boldsymbol{y}^2}\boldsymbol{\Lambda} + \frac{\partial^2 H_m}{\partial \boldsymbol{y}\partial \boldsymbol{z}}\mathbf{W}\Big) + \chi_i\frac{\partial \mathbf{w}^\top}{\partial \xi_i}\Big(\frac{\partial^2 H_m}{\partial \boldsymbol{z}\partial \boldsymbol{y}}\boldsymbol{\Lambda} + \frac{\partial^2 H_m}{\partial \boldsymbol{z}^2}\mathbf{W}\Big)+$$

$$\boldsymbol{\lambda}^\top\boldsymbol{\Lambda}^\top\Big(\frac{\partial^2 H_m}{\partial \boldsymbol{y}^2}\boldsymbol{\Lambda} + \frac{\partial^2 H_m}{\partial \boldsymbol{y}\partial \boldsymbol{z}}\mathbf{W}\Big) + \boldsymbol{\lambda}^\top\mathbf{W}^\top\Big(\frac{\partial^2 H_m}{\partial \boldsymbol{z}\partial \boldsymbol{y}}\boldsymbol{\Lambda} + \frac{\partial^2 H_m}{\partial \boldsymbol{z}^2}\mathbf{W}\Big)+$$

$$\boldsymbol{\mu}^\top\mathbf{V}^\top\Big(\frac{\partial^2 H_m}{\partial \boldsymbol{y}^2}\boldsymbol{\Lambda} + \frac{\partial^2 H_m}{\partial \boldsymbol{y}\partial \boldsymbol{z}}\mathbf{W}\Big). \tag{8D}$$

Since $\partial H_m / \partial \mathbf{y}$ is the row vector with the components $\partial H_m / \partial y_i$, we have

$$\frac{\partial H_m}{\partial \mathbf{y}}(\boldsymbol{\lambda}^\top \mathbf{R}) = \boldsymbol{\lambda}^\top \big(\frac{\partial H_m}{\partial y_i}\mathbf{R}_i\big).$$

Recall that the matrices $\mathbf{R}_i$ and Φ_4 are symmetric. From this and formula (4.9d), we conclude that

$$\frac{\partial H_m}{\partial \mathbf{y}}(\boldsymbol{\lambda}^\top \mathbf{R}) + \boldsymbol{\lambda}^\top \boldsymbol{\Lambda}^\top \Big(\frac{\partial^2 H_m}{\partial \boldsymbol{y}^2}\boldsymbol{\Lambda} + \frac{\partial^2 H_m}{\partial \boldsymbol{y}\partial \boldsymbol{z}}\mathbf{W}\Big)$$

$$+ \boldsymbol{\lambda}^\top \mathbf{W}^\top \Big(\frac{\partial^2 H_m}{\partial \boldsymbol{z}\partial \boldsymbol{y}}\boldsymbol{\Lambda} + \frac{\partial^2 H_m}{\partial \boldsymbol{z}^2}\mathbf{W}\Big) = \boldsymbol{\lambda}^\top \Phi_4^\top \qquad (9D)$$

Next, it follows from (4.9b) that

$$\frac{\partial H_m}{\partial \mathbf{y}}\frac{\partial \boldsymbol{\Lambda}}{\partial \xi_i} + \frac{\partial H_m}{\partial \mathbf{z}}\frac{\partial \mathbf{W}}{\partial \xi_i} + \frac{\partial}{\partial \xi_i}(\boldsymbol{\xi} + \mathbf{u})^\top \Big(\frac{\partial^2 H_m}{\partial \boldsymbol{x}\partial \boldsymbol{y}}\boldsymbol{\Lambda} + \frac{\partial^2 H_m}{\partial \boldsymbol{x}\partial \boldsymbol{z}}\mathbf{W}\Big) + \qquad (10D)$$

$$\frac{\partial \mathbf{v}^\top}{\partial \xi_i}\Big(\frac{\partial^2 H_m}{\partial \boldsymbol{y}^2}\boldsymbol{\Lambda} + \frac{\partial^2 H_m}{\partial \boldsymbol{y}\partial \boldsymbol{z}}\mathbf{W}\Big) + \frac{\partial \mathbf{w}^\top}{\partial \xi_i}\Big(\frac{\partial^2 H_m}{\partial \boldsymbol{z}\partial \boldsymbol{y}}\boldsymbol{\Lambda} + \frac{\partial^2 H_m}{\partial \boldsymbol{z}^2}\mathbf{W}\Big) = \frac{\partial \Phi_2^\top}{\partial \xi_i}.$$

We also have

$$\frac{\partial H_m}{\partial \mathbf{y}}\boldsymbol{\Lambda}\boldsymbol{\Gamma} + \frac{\partial H_m}{\partial \mathbf{z}}\mathbf{W}\boldsymbol{\Gamma} = \Phi_2^\top \boldsymbol{\Gamma}. \qquad (11D)$$

Substituting (9D)-(11D) into (8D) we obtain

$$\delta \Phi_2^\top = \frac{\partial H_m}{\partial \mathbf{y}}\mathbf{V}\nabla_\xi(\mathbf{J}\boldsymbol{\lambda}) + \boldsymbol{\mu}^\top \mathbf{V}^\top \Big(\frac{\partial^2 H_m}{\partial \boldsymbol{y}^2}\boldsymbol{\Lambda} + \frac{\partial^2 H_m}{\partial \boldsymbol{y}\partial \boldsymbol{z}}\mathbf{W}\Big) +$$

$$\chi_i \frac{\partial \Phi_2^\top}{\partial \xi_i} + \Phi_2^\top \boldsymbol{\Gamma} + \boldsymbol{\lambda}^\top \Phi_4^\top \qquad (12D)$$

Expression (4.25) for the matrix $\mathbf{T}$ implies

$$\boldsymbol{\mu}^\top \mathbf{V}^\top \Big(\frac{\partial^2 H_m}{\partial \boldsymbol{y}^2}\boldsymbol{\Lambda} + \frac{\partial^2 H_m}{\partial \boldsymbol{y}\partial \boldsymbol{z}}\mathbf{W}\Big) = \boldsymbol{\mu}^\top \boldsymbol{T}^\top. \qquad (13D)$$

On the other hand, formula (4.9c) for the operator Φ_3 yields

$$\frac{\partial H_m}{\partial \mathbf{y}}\mathbf{V}\nabla_\xi(\mathbf{J}\boldsymbol{\lambda}) = \boldsymbol{\omega}^\top \nabla_\xi(\mathbf{J}\boldsymbol{\lambda}) + \Phi_3^\top \nabla_\xi(\mathbf{J}\boldsymbol{\lambda}) =$$

$$(\boldsymbol{\partial}(\mathbf{J}\boldsymbol{\lambda}))^\top + \Big((\mathbf{J}\boldsymbol{\lambda})'_\xi \Phi_3\Big)^\top = (\mathbf{J}\boldsymbol{\partial}\boldsymbol{\lambda})^\top + \Big((\mathbf{J}\boldsymbol{\lambda})'_\xi \Phi_3\Big)^\top.$$

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Substituting this identity along with (13D) into (12D) we arrive at desired representation (4.23b) for $\delta\Phi_2$.

Our next task is to prove identity (4.23c) for $\delta\Phi_3$. It follows from identity (3.19e) in the Second Structural Theorem that

$$\delta\mathbf{V} \,=\, -\mathbf{V}\nabla_\xi\boldsymbol{\chi} + \chi_i\frac{\partial\mathbf{V}}{\partial\xi_i}$$

Substituting this identity along with (2D) into formula (1Dc) for $\delta\Phi_3$ we obtain

$$\delta\Phi_3^\top = -\frac{\partial H_m}{\partial\mathbf{y}}\mathbf{V}\nabla_\xi\boldsymbol{\chi} + \chi_i\frac{\partial H_m}{\partial\mathbf{y}}\frac{\partial\mathbf{V}}{\partial\xi_i}+$$
$$\chi_i\frac{\partial}{\partial\xi_i}(\boldsymbol{\xi}+\mathbf{u})^\top\frac{\partial^2 H_m}{\partial\mathbf{x}\partial\mathbf{y}} + \chi_i\frac{\partial\mathbf{v}^\top}{\partial\xi_i}\frac{\partial^2 H_m}{\partial\mathbf{y}^2} + \chi_i\frac{\partial\mathbf{w}^\top}{\partial\xi_i}\frac{\partial^2 H_m}{\partial\mathbf{z}\partial\mathbf{y}}+$$
$$\boldsymbol{\mu}^\top\mathbf{V}^\top\frac{\partial^2 H_m}{\partial\mathbf{y}^2}\mathbf{V} + \boldsymbol{\lambda}^\top\boldsymbol{\Lambda}^\top\frac{\partial^2 H_m}{\partial\mathbf{y}^2}\mathbf{V} + \boldsymbol{\lambda}^\top\mathbf{W}^\top\frac{\partial^2 H_m}{\partial\mathbf{z}\partial\mathbf{y}}\mathbf{V}. \qquad (14D)$$

It follows from formula (4.9c) for Φ_3 that

$$-\frac{\partial H_m}{\partial\mathbf{y}}\mathbf{V}\nabla_\xi\boldsymbol{\chi} = -\boldsymbol{\omega}^\top\nabla_\xi\boldsymbol{\chi} - \Phi_3^\top\nabla_\xi\boldsymbol{\chi} = -(\boldsymbol{\partial}\chi)^\top - (\boldsymbol{\chi}'_\xi\Phi_3)^\top. \qquad (15D)$$

Next, we have

$$\chi_i\frac{\partial H_m}{\partial\mathbf{y}}\frac{\partial\mathbf{V}}{\partial\xi_i} + \chi_i\frac{\partial}{\partial\xi_i}(\boldsymbol{\xi}+\mathbf{u})^\top\frac{\partial^2 H_m}{\partial\mathbf{x}\partial\mathbf{y}}+$$
$$\chi_i\frac{\partial\mathbf{v}^\top}{\partial\xi_i}\frac{\partial^2 H_m}{\partial\mathbf{y}^2} + \chi_i\frac{\partial\mathbf{w}^\top}{\partial\xi_i}\frac{\partial^2 H_m}{\partial\mathbf{z}\partial\mathbf{y}} = \chi_i\frac{\partial\Phi_3^\top}{\partial\xi_i} \qquad (16D)$$

On the other hand, formulae (4.25) for $\mathbf{S}$ and $\mathbf{T}$ yield

$$\boldsymbol{\mu}^\top\mathbf{V}^\top\frac{\partial^2 H_m}{\partial\mathbf{y}^2}\mathbf{V}+\boldsymbol{\lambda}^\top\boldsymbol{\Lambda}^\top\frac{\partial^2 H_m}{\partial\mathbf{y}^2}\mathbf{V}+\boldsymbol{\lambda}^\top\mathbf{W}^\top\frac{\partial^2 H_m}{\partial\mathbf{z}\partial\mathbf{y}}\mathbf{V} = (\mathbf{S}\boldsymbol{\mu})^\top+(\mathbf{T}^\top\boldsymbol{\lambda})^\top. \ (17D)$$

Substituting (15D)-(17D) into (14D) we arrive at the desired identity (1Dc).

It remains to prove representation (1Dd) for $\delta\Phi_4$. Recall that this relation includes the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, given by (1Df), and the matrices $\mathbf{P}^{(\tau)}$, $\boldsymbol{\tau} = \mathbf{x},\mathbf{y},\mathbf{z}$ given by (1Dg). Substituting representations (7D) into (1Df) we obtain

$$\mathbf{A} = \boldsymbol{\Lambda}^\top\frac{\partial^2 H_m}{\partial\mathbf{y}^2}\boldsymbol{\Lambda} + \chi_i\frac{\partial\boldsymbol{\Lambda}^\top}{\partial\xi_i}\frac{\partial^2 H_m}{\partial\mathbf{y}^2}\boldsymbol{\Lambda} + (\mathbf{J}\boldsymbol{\lambda})'_\xi\mathbf{V}^\top\frac{\partial^2 H_m}{\partial\mathbf{y}^2}\boldsymbol{\Lambda} + (\boldsymbol{\lambda}^\top\mathbf{R})^\top\frac{\partial^2 H_m}{\partial\mathbf{y}^2}\boldsymbol{\Lambda},$$
$$(18D)$$

$$\mathbf{B} = \chi_i \frac{\partial \mathbf{W}^\top}{\partial \xi_i} \frac{\partial^2 H_m}{\partial \mathbf{z} \mathbf{y}} \boldsymbol{\Lambda} + \boldsymbol{\Gamma}^\top \frac{\partial^2 H_m}{\partial \mathbf{z} \mathbf{y}} \boldsymbol{\Lambda} + \tag{19D}$$

$$\mathbf{W}^\top \frac{\partial^2 H_m}{\partial \mathbf{z} \mathbf{y}} \mathbf{V} \nabla_\xi (\mathbf{J} \boldsymbol{\lambda}) + \chi_i \mathbf{W}^\top \frac{\partial^2 H_m}{\partial \mathbf{z} \mathbf{y}} \frac{\partial \boldsymbol{\Lambda}}{\partial \xi_i} + \mathbf{W}^\top \frac{\partial^2 H_m}{\partial \mathbf{z} \mathbf{y}} (\boldsymbol{\lambda}^\top \mathbf{R}) + \mathbf{W}^\top \frac{\partial^2 H_m}{\partial \mathbf{z} \mathbf{y}} \boldsymbol{\Lambda} \boldsymbol{\Gamma},$$

$$\mathbf{C} = \chi_i \frac{\partial \mathbf{W}^\top}{\partial \xi_i} \frac{\partial^2 H_m}{\partial \mathbf{z} \mathbf{y}} \mathbf{W} + \boldsymbol{\Gamma}^\top W^\top \frac{\partial^2 H_m}{\partial \mathbf{z} \mathbf{y}} \mathbf{W} \tag{20D}$$

Next, expression (1Dg) for $\mathbf{P}^{(\tau)}$ and representations (2D) implies

$$\delta \mathbf{u}^\top \mathbf{P}^{(x)} + \delta \mathbf{v}^\top \mathbf{P}^{(y)} + \delta \mathbf{w}^\top \mathbf{P}^{(z)} = \chi_i \Big(\frac{\partial \mathbf{u}^\top}{\partial \xi_i} \mathbf{P}^{(x)} + \frac{\partial \mathbf{v}^\top}{\partial \xi_i} \mathbf{P}^{(y)} + \frac{\partial \mathbf{w}^\top}{\partial \xi_i} \mathbf{P}^{(z)} \Big) +$$

$$\boldsymbol{\mu}^\top \mathbf{P}^{(y)} + \boldsymbol{\lambda}^\top (\boldsymbol{\Lambda}^\top \mathbf{P}^{(y)} + \mathbf{W}^\top \mathbf{P}^{(z)}) \tag{21D}$$

Substituting (18D) (21D) into (1Dd) and recalling the representation

$$\delta \mathbf{R}_i = \frac{\partial}{\partial \xi_i} (\mathbf{J} \boldsymbol{\Gamma}) + \mathbf{R}_i \boldsymbol{\Gamma} + (\mathbf{R}_i \boldsymbol{\Gamma})^\top + \chi_i \frac{\partial}{\partial \xi_i} \mathbf{R}_i$$

in the second structural Theorem 3.2 we arrive at the identity

$$\delta \Phi_4 = \frac{\partial H_m}{\partial y_i} V_{ik} \frac{\partial}{\partial \xi_k} (\mathbf{J} \boldsymbol{\Gamma}) + \mathbf{Q} \boldsymbol{\Gamma} + \boldsymbol{\Gamma}^\top \mathbf{Q}^\top +$$

$$\chi_k \mathbf{N}_k + \mathbf{U} + \mathbf{U}^\top + \mathbf{K} + \mathbf{K}^\top + \boldsymbol{\Sigma}. \tag{4.22}$$

Here

$$\mathbf{Q} = \frac{\partial H_m}{\partial y_i} \mathbf{R}_i + \boldsymbol{\Lambda}^\top \frac{\partial^2 H_m}{\partial \mathbf{y}^2} \boldsymbol{\Lambda} +$$

$$\mathbf{W}^\top \frac{\partial^2 H_m}{\partial \mathbf{z} \partial \mathbf{y}} \boldsymbol{\Lambda} + \Big(\mathbf{W}^\top \frac{\partial^2 H_m}{\partial \mathbf{z} \partial \mathbf{y}} \boldsymbol{\Lambda} \Big)^\top + \mathbf{W}^\top \frac{\partial^2 H_m}{\partial \mathbf{z}^2} \mathbf{W} = \boldsymbol{\Omega} + \Phi_4, \tag{4.23}$$

$$\mathbf{N}_k = \frac{\partial H_m}{\partial y_i} \frac{\partial \mathbf{R}_i}{\partial \xi_k} + \boldsymbol{\Lambda}^\top \frac{\partial^2 H_m}{\partial \mathbf{y}^2} \frac{\partial \boldsymbol{\Lambda}}{\partial \xi_k} + \frac{\partial \boldsymbol{\Lambda}^\top}{\partial \xi_k} \frac{\partial^2 H_m}{\partial \mathbf{y}^2} \boldsymbol{\Lambda} +$$

$$\mathbf{W}^\top \frac{\partial^2 H_m}{\mathbf{z} \partial \mathbf{y}} \frac{\partial \boldsymbol{\Lambda}}{\partial \xi_k} + \frac{\partial \mathbf{W}^\top}{\partial \xi_k} \frac{\partial^2 H_m}{\partial \mathbf{z} \partial \mathbf{y}} \boldsymbol{\Lambda} +$$

$$\Big(\mathbf{W}^\top \frac{\partial^2 H_m}{\partial \mathbf{z} \partial \mathbf{y}} \frac{\partial \boldsymbol{\Lambda}}{\partial \xi_k} + \frac{\partial \mathbf{W}^\top}{\partial \xi_k} \frac{\partial^2 H_m}{\partial \mathbf{z} \mathbf{y}} \boldsymbol{\Lambda} \Big)^\top + \tag{4.24}$$

$$\mathbf{W}^\top \frac{\partial^2 H_m}{\partial \mathbf{z}^2} \frac{\partial \mathbf{W}}{\partial \xi_k} + \frac{\partial \mathbf{W}^\top}{\partial \xi_k} \frac{\partial^2 H_m}{\partial \mathbf{z}^2} \mathbf{W} +$$

$$+ \frac{\partial \mathbf{u}^\top}{\partial \xi_i} \mathbf{P}^{(x)} + \frac{\partial \mathbf{v}^\top}{\partial \xi_i} \mathbf{P}^{(y)} + \frac{\partial \mathbf{w}^\top}{\partial \xi_i} \mathbf{P}^{(z)} = \frac{\partial \Phi_4}{\partial \xi_k}. \tag{4.25}$$

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Next notice that in view of (4.9c),

$$\frac{\partial}{\partial \xi_i}(\mathbf{J}\mathbf{\Gamma}) = \boldsymbol{\partial}(\mathbf{J}\mathbf{\Gamma}) + \Phi_{3,k}\frac{\partial}{\partial \xi_k}(\mathbf{J}\mathbf{\Gamma}).$$

Substituting this relation along with (4.23), (4.25) into (4.22) we arrive at the identity

$$\delta\Phi_4 = \boldsymbol{\partial}(\mathbf{J}\mathbf{\Gamma}) + \mathbf{\Omega}\mathbf{\Gamma} + \mathbf{\Gamma}^\top\mathbf{\Omega}^\top +$$

$$\Phi_4\mathbf{\Gamma} + \mathbf{\Gamma}^\top\Phi_4^\top + \chi_k\frac{\partial\Phi_4}{\partial\xi_k} + \Phi_{3,k}\frac{\partial}{\partial\xi_k}(\mathbf{J}\mathbf{\Gamma}) +$$

$$\mathbf{U} + \mathbf{U}^\top + \mathbf{K} + \mathbf{K}^\top + \mathbf{\Sigma}. \tag{4.26}$$

Notice that $\mathbf{U}$ is the matrix-valued linear form of $\partial\lambda_i/\partial\xi_j$. Hence it admits the representation

$$\mathbf{U} + \mathbf{U}^\top = \frac{\partial\lambda_i}{\partial\xi_j}\mathbf{U}_{ij}, \tag{4.27}$$

where $\mathbf{U}_{ij}$ are symmetric matrix-valued functions. Next, $\mathbf{K}$ and $\mathbf{\Sigma}$ are matrix-valued linear form of λ_i and μ_j. Hence there are symmetric matrix-valued functions $\mathbf{E}_i$ and $\mathbf{K}_i$ such that

$$\mathbf{K} + \mathbf{K}^\top + \mathbf{\Sigma} = \mu_i\mathbf{E}_i + \lambda_i\mathbf{K}_i. \tag{4.28}$$

Substituting (4.27) and (4.28) into (4.26) we obtain the desired representation (4.24d).

E   Proof of Theorems 5.1 and 8.1

E.1   Proof of Theorem 5.1

This section is devoted to the proof of solvability of the following problem

$$\boldsymbol{\partial}\psi_0 + \delta q + \delta p \cdot (w_1 - \alpha) + \frac{1}{2}\delta M(w_1 - \alpha)^2 = F_1, \tag{1Ea}$$

$$\mathbf{J}\boldsymbol{\partial}\boldsymbol{\lambda} + \mathbf{T}\boldsymbol{\mu} + \mathbf{W}^\top\delta\mathbf{p} + \delta M(w_1 - \alpha)\mathbf{W}^\top\mathbf{e}_1 = F_2, \quad \boldsymbol{\mu} = \nabla\psi_0 + \delta\boldsymbol{\beta} \tag{1Eb}$$

$$-\boldsymbol{\partial}\boldsymbol{\chi} + \mathbf{S}\boldsymbol{\mu} + \mathbf{T}^\top\boldsymbol{\lambda} = F_3, \tag{1Ec}$$

$$\boldsymbol{\partial}(\mathbf{J}\mathbf{\Gamma}) + \mathbf{\Omega}\mathbf{\Gamma} + (\mathbf{\Omega}\mathbf{\Gamma})^\top + \mathbf{U}_{ij}\frac{\partial\lambda_i}{\partial\xi_j} + \mu_i\mathbf{E}_i + \lambda_i\mathbf{K}_i + \tag{1Ed}$$

$$+\mathbf{W}^\top\delta\mathbf{M}\mathbf{W} = F_4, \quad \Gamma_{11} = -\Gamma_{22},$$

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$$\frac{1}{(2\pi)^{n-1}} \int_{\mathbb{T}^{n-1}} \left[\psi_0 - \delta\boldsymbol{\beta} \cdot \mathbf{u} + w_2 \big(\mathbf{W}\boldsymbol{\lambda}\big)_1 \right] d\xi = f_1 \qquad \text{(1Ee)}$$

$$\frac{1}{(2\pi)^{n-1}} \int_{\mathbb{T}^{n-1}} \mathbf{V}^{-\top} \boldsymbol{\chi} \, d\xi = f_3, \qquad \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{T}^{n-1}} (\mathbf{W}\boldsymbol{\Gamma})_{12} \, d\xi = f_4 \qquad \text{(1Ef)}$$

$$\frac{1}{(2\pi)^{n-1}} \int_{\mathbb{T}^{n-1}} \Big((\mathbf{W}\boldsymbol{\lambda})_1 + \boldsymbol{\chi} \cdot \nabla w_1 \Big) \, d\boldsymbol{\xi} = f_5. \qquad \text{(1Eg)}$$

The following theorem constitutes the existence and uniqueness of solutions to problem (1E)

Theorem E.1. *Let a fixed $\sigma \in [1/2, 1]$, $d \geq 2$, and the matrix $\mathbf{K}_0 = \mathbf{S}_0 - \mathbf{t}_0 \otimes \mathbf{t}_0$ given by (2.11), satisfies the condition $\det \mathbf{K}_0 \neq 0$. Furthermore, assume that*

$$\|w_1 - \alpha\|_{\sigma,0} + \|\mathbf{W} - \mathbf{I}\|_{\sigma_0} + \|\mathbf{u}' - \mathbf{I}\|_{\sigma,0} \leq cr \qquad \text{(2E)}$$

and

$$\|\mathbf{S} - \mathbf{S}_0\|_{\sigma,0} + \|\mathbf{T} - \mathbf{T}_0\|_{\sigma,0} \leq c(r + |\varepsilon|), \qquad \text{(3E)}$$

$$\|\mathbf{U}_{ij}\|_{\sigma,0} + \|\mathbf{E}_i\|_{\sigma,0} + \|\mathbf{K}_i\|_{\sigma,0} \leq c. \qquad \text{(4E)}$$

Then there are $\varepsilon_0 > 0$ and $r_0 > 0$ with the following properties. For every

$$r \leq r_0 \quad |\varepsilon| \leq \varepsilon_0, \quad ,$$

$$0 \leq \sigma_0 < \sigma_1 < \sigma, \quad 1/4 \geq \sigma_1, \sigma \in [1/4, 1),$$

and for all

$$\mathbf{F} = (F_1, F_2, F_3, F_4, f_1, f_3, f_4, f_5) \in \mathcal{A}_{\sigma_1,0} \times \mathcal{A}^2_{\sigma_1,0} \times \mathcal{A}^{n-1}_{\sigma_1,0} \times \mathcal{A}^4_{\sigma_1,0} \times \mathbb{C} \times \mathbb{C}^{n-1} \mathbb{C} \times \mathbb{C},$$

with $F_4 = F_4^\top$, problem (1E) has a unique solution

$$(\psi_0, \boldsymbol{\lambda}, \boldsymbol{\chi}, \boldsymbol{\Gamma}, \delta\boldsymbol{\beta}, q, p, \delta M) \in \mathcal{A}_{\sigma_0,0} \times \mathcal{A}^2_{\sigma_0,0} \times \mathcal{A}^{n-1}_{\sigma,0} \times \mathcal{A}^4_{\sigma_0,0} \times \mathbb{C}^{n-1} \times \mathbb{C}^3.$$

This solution admits the estimate

$$\|(\psi_0, \boldsymbol{\lambda}, \boldsymbol{\chi}, \boldsymbol{\Gamma})\|_{\sigma_0,0} + |(\delta\boldsymbol{\beta}, q, p, \delta M)| \leq c(\sigma_1 - \sigma_0)^{-8n-12} \|\mathbf{F}\|_{\sigma_1,0} \qquad \text{(5E)}$$

where the constant c is independent of ε_0, r_0, and σ_i.

Notice that identity $\mathbf{V} = (\mathbf{I} + \mathbf{u}')^{-\top}$ implies that

$$\|\mathbf{V}^{-\top} - \mathbf{I}\|_{\sigma_0} \leq cr.$$

The rest of the section is devoted to the proof of this theorem. Our strategy is the following. First we prove the existence and uniqueness of solutions to a triangle truncated problem. Then we prove the solvability of equations (5.6) using the contraction mapping principle. Our considerations are based on the following existence and uniqueness results for model differential equations with constant coefficients

Equations with constant coefficients   Recall the denotations

$$\overline{\boldsymbol{\mu}} = \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{T}^{n-1}} \boldsymbol{\mu}(\boldsymbol{\xi})\, d\boldsymbol{\xi}, \quad \boldsymbol{\mu}^* = \boldsymbol{\mu} - \overline{\boldsymbol{\mu}}$$

Lemma E.2. *For every $g \in \mathcal{A}_{\sigma_1,0}$ with $\overline{g} = 0$, the equations*

$$\boldsymbol{\partial}\psi_0 = g, \quad \overline{\psi}_0 = 0 \tag{6E}$$

have a unique solution $\psi_0 \in \mathcal{A}_{\sigma_0,0}$ such that

$$\|\psi_0\|_{\sigma_0,0} \leq c(\sigma_1 - \sigma_0)^{-\tau}\|g\|_{\sigma_1,0}, \quad |\psi_0|_s \leq c|g|_{s+\tau}. \tag{7E}$$

where $\tau = n + 1$ and $0 \leq \sigma_0 < \sigma_1$ are arbitrary numbers.

Proof. Substituting the decomposition

$$\psi_0(\xi) = \sum_{\mathbf{s} \in \mathbb{Z}^{n-1} \setminus \{0\}} \widehat{\psi_0}(\mathbf{s})e^{i\mathbf{s}\boldsymbol{\xi}}, \quad \widehat{\psi_0}(\mathbf{s}) = (2\pi)^{\frac{1-n}{2}} \int_{\mathbb{T}^{n-1}} \psi_0(\boldsymbol{\xi})e^{-i\mathbf{s}\boldsymbol{\xi}}\, d\boldsymbol{\xi},$$

into equation (65Ea) we can rewrite this equation into the equivalent form

$$\widehat{\psi_0}(\mathbf{s}) = (i\,\boldsymbol{\omega} \cdot \mathbf{s})^{-1}\, \widehat{g}(\mathbf{s}) \ \ \text{for} \ \ \mathbf{s} \in \mathbb{Z}^{n-1} \setminus \{0\}, \quad \widehat{\psi_0}(0) = 0 \tag{8E}$$

The diophantine condition implies

$$|(\boldsymbol{\omega} \cdot \mathbf{s})^{-1}| \leq c|\mathbf{s}|^{-n}, \tag{9E}$$

which gives

$$|\widehat{\psi_0}(\mathbf{s})| \leq (1 + |\mathbf{s}|)^n |\widehat{g}(\mathbf{s})| \tag{10E}$$

It is well-known that for every $\sigma \in [0,1)$ and $\sigma' \in (0,\sigma)$, and for every measurable function $g : \mathbb{T}^{n-1} \to \mathbb{C}$, we have

$$\sup_{\mathbf{s} \in \mathbb{Z}^{n-1}} e^{\sigma|\mathbf{s}|}|\widehat{g}(\mathbf{s})| \leq \|g\|_{\sigma,0}, \quad \|g\|_{\sigma',0} \leq \frac{c}{\sigma - \sigma'} \sup_{\mathbf{s} \in \mathbb{Z}^{n-1}} e^{\sigma|\mathbf{s}|}|\widehat{g}(\mathbf{s})|,$$

where c is independent of g, σ, and σ'. ¿From this and (10E) that

$$e^{(\sigma_0+\sigma_1)|\mathbf{s}|/2}|\widehat{\psi_0}(\mathbf{s})| \leq c(1+|\mathbf{s}|)^n e^{(\sigma_0+\sigma_1)|\mathbf{s}|/2}|\widehat{g}(\mathbf{s})| \leq$$
$$c(\sigma_1-\sigma_0)^{-n} e^{\sigma_1|\mathbf{s}|}|\widehat{g}(\mathbf{s})| \leq c(\sigma_1-\sigma_0)^{-\tau}\|g\|_{\sigma_1,0}. \tag{11E}$$

It follows that

$$\|\psi_0\|_{\sigma_0,0} \leq \frac{c}{\sigma_1 - \sigma_0} \sup_{\mathbf{s} \in \mathbb{Z}^{n-1}} e^{(\sigma_0+\sigma_1)|\mathbf{s}|/2}|\widehat{\psi_0}(\mathbf{s})| \leq c(\sigma_1-\sigma_0)^{-\tau}\|g\|_{\sigma_1,0}.$$

Thus we get the first estimate in (7E). The second obviously follows from (10E) and the definition of the Sobolev space H_r. $\square$

Lemma E.3. *Let $0 \leq \sigma_0 < \sigma_1$. Let $\mathbf{H} \in \mathcal{A}_{\sigma_1,0}$ satisfies the condition $\overline{\mathbf{H}} = 0$. Then the equation*

$$\mathbf{J}\,\partial\boldsymbol{\lambda} + \boldsymbol{\Omega}\boldsymbol{\lambda} = \mathbf{H} \tag{12E}$$

has a unique solution $\boldsymbol{\lambda} \in \mathcal{A}_{\sigma_0,0}$, satisfying the condition $\overline{\boldsymbol{\lambda}} = 0$. This solution admits the estimates

$$\|\boldsymbol{\lambda}\|_{\sigma_0,0} \leq c(\sigma_1-\sigma_0)^{-2n-3}\,\|\mathbf{H}\|_{\sigma_1,0}. \tag{13E}$$

$$|\boldsymbol{\lambda}|_s \leq c\,|\mathbf{H}|_{s+2n+3}. \tag{14E}$$

Proof. Equation (12E) is equivalent to the following system of the linear algebraic equations for the Fourier coefficients $\widehat{\boldsymbol{\lambda}}(\mathbf{s})$, $\mathbf{s} \in \mathbb{Z}^{n-1}$, of $\boldsymbol{\lambda}$,

$$i(\boldsymbol{\omega} \cdot \mathbf{s})\,\mathbf{J}\,\widehat{\boldsymbol{\lambda}}(\mathbf{s}) + \boldsymbol{\Omega}\,\widehat{\boldsymbol{\lambda}}(\mathbf{s}) = \widehat{\mathbf{H}}(\mathbf{s}), \quad \mathbf{s} \in \mathbb{Z}^{n-1} \setminus \{0\}, \quad \widehat{\boldsymbol{\lambda}}(0) = 0.$$

This equality along with expressions (2.6) and (2.9) for the matrices $\mathbf{J}$ and $\boldsymbol{\Omega}$ implies

$$\widehat{\lambda_1}(\mathbf{s}) = -\big(k + (\boldsymbol{\omega} \cdot \mathbf{s})^2\big)^{-1}\big(\widehat{H_1}(\mathbf{s}) - i(\boldsymbol{\omega} \cdot \mathbf{s})\widehat{H_2}(\mathbf{s})\big), \, \widehat{\lambda_1}(\mathbf{s}) = \widehat{H_2}(\mathbf{s}) + i(\boldsymbol{\omega} \cdot \mathbf{s})\widehat{\lambda_1}(\mathbf{s}).$$

Recall that $\mathrm{Re}\,k \geq 0$. From this and the diophantine estimate (9E) we conclude that

$$|\widehat{\boldsymbol{\lambda}}(\mathbf{s})| \leq c|\mathbf{s}|^{2n+2}\,|\widehat{\mathbf{H}}(\mathbf{s})|, \quad \mathbf{s} \in \mathbb{Z}^{n-1}.$$

Thus we get

$$\|\boldsymbol{\lambda}\|_{\sigma_0,0} \le c(\sigma_1 - \sigma_0)^{-1} \sup_{\mathbf{s}\in\mathbb{Z}^{n-1}} e^{(\sigma_0+\sigma_1)|\mathbf{s}|/2} \, |\widehat{\boldsymbol{\lambda}}(\mathbf{s})|$$

$$c(\sigma_1 - \sigma_0)^{-2n-3} \sup_{\mathbf{s}\in\mathbb{Z}^{n-1}} e^{(\sigma_1|\mathbf{s}|} \, |\widehat{\mathbf{H}}(\mathbf{s})| \le c(\sigma_1 - \sigma_0)^{-2n-3} \, \|\mathbf{H}\|_{\sigma_1,0},$$

and the lemma follows. □

Now we consider the matrix differential equation

$$\boldsymbol{\partial}(\mathbf{J}\boldsymbol{\Gamma}) + \boldsymbol{\Omega}\boldsymbol{\Gamma} + (\boldsymbol{\Omega}\boldsymbol{\Gamma})^\top + \delta\mathbf{M} = \mathbf{G}, \tag{15E}$$

$$\Gamma_{11} = -\Gamma_{22}, \quad \int_{\mathbb{T}^{n-1}} \Gamma_{12}\, d\xi = f, \tag{16E}$$

Her $\delta\mathbf{M} = \delta M \mathrm{diag}(1,0)$ is the unknown matrix.

Lemma E.4. *For all* $(\mathbf{G}, f) \in A_{\sigma_1,0} \times \mathbb{C}$ *and for all* $0 \le \sigma_0 < \sigma_1$*, equations* (15E)-(16E) *have a unique solution* $(\boldsymbol{\Gamma}, \delta M) \in \mathcal{A}_{\sigma_0,0}^4 \times \mathbb{C}$. *This solution admits the estimate*

$$\|\boldsymbol{\Gamma}\|_{\sigma_0,0} + |\delta\mathbf{M}| \le c(\sigma_1 - \sigma_0)^{-3n-2}(\|\mathbf{G}\|_{\sigma_1,0} + |f|). \tag{17E}$$

Proof. Rewrite equations (15E)-(16E) in the form

$$\boldsymbol{\Omega}\overline{\boldsymbol{\Gamma}} + (\boldsymbol{\Omega}\overline{\boldsymbol{\Gamma}})^\top + \delta\mathbf{M} = \overline{\mathbf{G}},$$
$$\overline{\Gamma}_{11} = -\overline{\Gamma}_{22}, \quad \overline{\Gamma}_{12} = f_4, \quad \delta\mathbf{M} = \mathrm{diag}\,(\delta M, 0). \tag{18E}$$

$$\boldsymbol{\partial}(\mathbf{J}\boldsymbol{\Gamma}^*) + \boldsymbol{\Omega}\boldsymbol{\Gamma}^* + (\boldsymbol{\Omega}\boldsymbol{\Gamma}^*)^\top = \mathbf{G}^*, \quad \Gamma_{11}^* = -\Gamma_{22}^*. \tag{19E}$$

The unique solution to equations (18E) is given by

$$\overline{\boldsymbol{\Gamma}} = \begin{pmatrix} -\overline{G}_{22}/2, & f_4 \\ kf_4 + \overline{G}_{21} & \overline{G}_{22}/2 \end{pmatrix}, \quad \delta\mathbf{M} = \begin{pmatrix} \overline{G}_{11} - k\overline{G}_{22}, & 0 \\ 0 & 0 \end{pmatrix}.$$

It follows that

$$|\overline{\boldsymbol{\Gamma}}| + |\delta M| \le c|\mathbf{G}|. \tag{20E}$$

Next, using the Fourier transform we can write equation (19E) in the form of the matrix equation

$$i(\boldsymbol{\omega}\cdot\boldsymbol{s})\mathbf{J}\widehat{\boldsymbol{\Gamma}^*}(\mathbf{s}) + \boldsymbol{\Omega}\widehat{\boldsymbol{\Gamma}^*}(\mathbf{s}) + \left(\boldsymbol{\Omega}\widehat{\boldsymbol{\Gamma}^*}(\mathbf{s})\right)^\top = \widehat{\mathbf{G}^*}(\mathbf{s}), \quad \widehat{\Gamma_{11}^*}(\mathbf{s}) = -\widehat{\Gamma_{22}^*}(\mathbf{s}) \text{ for } \mathbf{s} \in \mathbb{Z}^2\backslash\{0\},$$

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Recall that

$$\widehat{\boldsymbol{\Gamma}^*}(0) = \widehat{\boldsymbol{G}^*}(0) = 0. \tag{21E}$$

In it turn, this equation is equivalent to the system of linear algebraic equations

$$i(\boldsymbol{\omega} \cdot \mathbf{s})\,\widehat{\Gamma^*_{21}}(\mathbf{s}) + 2k\,\widehat{\Gamma^*_{22}}(\mathbf{s}) = \widehat{G^*_{11}}(\mathbf{s}),$$
$$i(\boldsymbol{\omega} \cdot \mathbf{s})\,\widehat{\Gamma^*_{22}}(\mathbf{s}) + \widehat{\Gamma^*_{21}} - k\,\widehat{\Gamma^*_{12}}(\mathbf{s}) = \widehat{G^*_{12}}(\mathbf{s}),$$
$$-i(\boldsymbol{\omega} \cdot \mathbf{s})\,\widehat{\Gamma^*_{12}}(\mathbf{s}) + 2\,\widehat{\Gamma^*_{22}}(\mathbf{s}) = \widehat{G^*_{22}}(\mathbf{s}),$$
$$\widehat{\Gamma^*_{11}}(\mathbf{s}) = -\widehat{\Gamma^*_{22}}(\mathbf{s}) \text{ for } \mathbf{s} \in \mathbb{Z}^2 \setminus \{0\}.$$

Straightforward calculations give

$$\widehat{\Gamma^*_{12}}(\mathbf{s}) = \frac{2}{(\boldsymbol{\omega} \cdot \mathbf{s})^2 + 4k}\left(\frac{1}{i(\boldsymbol{\omega} \cdot \mathbf{s})}\left(\widehat{G^*_{11}}(\mathbf{s}) - k\widehat{G^*_{22}}(\mathbf{s})\right) + i(\boldsymbol{\omega} \cdot \mathbf{s})\widehat{G^*_{22}}(\mathbf{s}) - \widehat{G^*_{12}}(\mathbf{s})\right)$$

and

$$\widehat{\Gamma^*_{22}}(\mathbf{s}) = \frac{1}{2}\widehat{G^*_{22}}(\mathbf{s}) + \frac{1}{2}i(\boldsymbol{\omega} \cdot \mathbf{s})\widehat{\Gamma^*_{12}}(\mathbf{s}),$$
$$\widehat{\Gamma^*_{21}}(\mathbf{s}) = \widehat{G^*_{12}}(\mathbf{s}) + k\widehat{\Gamma^*_{12}}(\mathbf{s}) - i(\boldsymbol{\omega} \cdot \mathbf{s})\widehat{\Gamma^*_{22}}(\mathbf{s})$$
$$\widehat{\Gamma^*_{11}}(\mathbf{s}) = -\widehat{\Gamma^*_{22}}(\mathbf{s}).$$

These identities and the diophantine estimate (9E) imply the estimate

$$|\widehat{\boldsymbol{\Gamma}^*}(\mathbf{s})| \leq c|\mathbf{s}|^{3n+1}|\widehat{\boldsymbol{G}^*}(\mathbf{s})| \text{ for } \mathbf{s} \in \mathbb{Z}^2.$$

Set $\varsigma = (\sigma_1 - \sigma_0)/3$. Arguing as in the proof of Lemmas E.3 and 17E we obtain

$$\|\boldsymbol{\Gamma}^*\|_{\sigma_0,0} \leq c\varsigma^{-1} \sup_{\mathbf{s}} e^{(\sigma_0+\varsigma)|\mathbf{s}|}|\widehat{\boldsymbol{\Gamma}^*}(\mathbf{s})| \leq$$
$$c\varsigma^{-1} \sup_{\mathbf{s}} |\mathbf{s}|^{3n+1}e^{(\sigma_0+\varsigma)|\mathbf{s}|}|\widehat{\boldsymbol{G}^*}(\mathbf{s})| \leq c\varsigma^{-3n-2} \sup_{\mathbf{s}} e^{(\sigma_0+\varsigma)|\mathbf{s}|}|\widehat{\boldsymbol{G}^*}(\mathbf{s})|$$
$$\leq c\varsigma^{-3n-2}\|\boldsymbol{G}^*\|_{\sigma_0+\varsigma,0}.$$

Combining this estimate with (20E) we arrive at

$$\|\boldsymbol{\Gamma}\|_{\sigma_0,0} + |\delta\mathbf{M}| \leq c\varsigma^{-3n-2}\|\boldsymbol{G}\|_{\sigma_0+\varsigma,0}. \tag{22E}$$

This completes the proof of the lemma.

$$\square$$

Truncated equations. Our next task is to prove the existence and uniqueness of analytic solutions to linear equations with variable coefficients which can be regarded as simplification of the main problem (1E). First we consider the shortened version of this problem which can be formulated as follows. It is necessary to find periodic functions ψ_0, $\boldsymbol{\lambda}$, $\boldsymbol{\chi}$ and parameters p, q and β satisfying the following equations.

$$\boldsymbol{\partial}\psi_0 + q = F_1, \tag{23Ea}$$

$$\mathbf{J}\boldsymbol{\partial}\boldsymbol{\lambda} + \boldsymbol{\Omega}\boldsymbol{\lambda} + \mathbf{T}\boldsymbol{\mu} + p\,\mathbf{e}_1 = F_2, \quad \boldsymbol{\mu} = \delta\boldsymbol{\beta} + \nabla\psi_0 \tag{23Eb}$$

$$-\boldsymbol{\partial}\boldsymbol{\chi} + \mathbf{S}\boldsymbol{\mu} + \mathbf{T}^\top\boldsymbol{\lambda} = F_3, \tag{23Ec}$$

$$\overline{\psi_0} = f_1, \quad \overline{\boldsymbol{\chi}} = f_3, \quad \overline{\lambda_1} = f_5. \tag{23Ed}$$

Proposition E.5. *Under the assumptions of Theorem E.1, there exist $r_0 > 0$, $\varepsilon_0 > 0$, and $c > 0$ such that for all $r \leq r_0$, $|\varepsilon| \leq \varepsilon_0$, and for all $(f_5, f_3, f_1) \in \mathbb{C}^{n+1}$, $F_i \in \mathcal{A}_{\sigma_1,0}$, and for all $0 \leq \sigma_0 < \sigma_1$, equations (23E) have a unique solution analytic solution. This solution admits the estimate*

$$\|\psi_0\|_{\sigma_0,0} + \|\boldsymbol{\lambda}\|_{\sigma_0,0} + \|\boldsymbol{\chi}\|_{\sigma_0,0} +$$
$$|p| + |q| + |\delta\boldsymbol{\beta}| \leq c(\sigma_1 - \sigma_0)^{-6n-8}(\|\mathbf{F}\|_{\sigma_1,0} + |\mathbf{f}|). \tag{24E}$$

Proof. We have

$$\boldsymbol{\lambda} = \overline{\boldsymbol{\lambda}} + \boldsymbol{\lambda}^*, \quad \boldsymbol{\chi} = f_3 + \boldsymbol{\chi}^* \tag{25E}$$

Substituting this decomposition into (23E) we obtain two system of equations

$$\overline{\psi_0} = f_1, \quad \overline{\boldsymbol{\chi}} = f_3, \quad \overline{\lambda_1} = f_5 \tag{26E}$$

$$\boldsymbol{\Omega}\overline{\boldsymbol{\lambda}} + \overline{\mathbf{T}}\boldsymbol{\beta} + p\,\mathbf{e}_1 = \overline{F_2} - \overline{\mathbf{T}\nabla\psi_0}, \tag{27E}$$

$$\overline{\mathbf{S}}\boldsymbol{\beta} + \overline{\mathbf{T}^\top}\overline{\boldsymbol{\lambda}} = \overline{F_3} - \overline{\mathbf{S}\nabla\psi_0} - \overline{\mathbf{T}^\top\boldsymbol{\lambda}^*}, \tag{28E}$$

and

$$\boldsymbol{\partial}\psi_0^* = F_1^*, \tag{29E}$$

$$\mathbf{J}\boldsymbol{\partial}\boldsymbol{\lambda}^* + \boldsymbol{\Omega}\boldsymbol{\lambda}^* + (\mathbf{T}\nabla\psi_0^*)^* = F_2^* - \mathbf{T}^*\boldsymbol{\beta}, \tag{30E}$$

$$-\boldsymbol{\partial}\boldsymbol{\chi}^* + (\mathbf{S}\nabla\psi_0^*)^* + (\mathbf{T}^\top\boldsymbol{\lambda}^*) = F_3^* - \mathbf{S}^*\boldsymbol{\beta} - (\mathbf{T}^\top)^*\overline{\boldsymbol{\lambda}}, \tag{31E}$$

The rest of the proof is based on the following auxiliary lemma.

Lemma E.6. . *Under the assumptions of Theorem 1E there exist $r_0 > 0$ and $\varepsilon_0 > 0$, $c > 0$ such that for all $r \le r_0$ and $|\varepsilon| \le \varepsilon_0$, and for given $\nabla\psi_0, \lambda^* \in H_0$, the system (26E)-(28E) has a unique solution, which admits the representation*

$$\boldsymbol{\beta} = \mathcal{M}(\nabla\psi_0^*, \boldsymbol{\lambda}^*) + \mathfrak{F}_\beta, \tag{32Ea}$$

$$\overline{\boldsymbol{\lambda}} = \mathcal{L}(\nabla\psi_0^*, \boldsymbol{\lambda}^*) + \mathfrak{F}_\lambda, \tag{32Eb}$$

$$p = \mathcal{P}(\nabla\psi_0^*, \boldsymbol{\lambda}^*) + \mathfrak{F}_p. \tag{32Ec}$$

Here the constant vectors $\mathfrak{F}_\beta$, $\mathfrak{F}_\lambda$, $\mathfrak{F}_p$ and the linear functionals $\mathcal{M}$, $\mathcal{L}$, $\mathcal{P}$ admit the estimates

$$|\mathfrak{F}_\beta| \le c(|\overline{F}_2| + |\overline{F}_3| + |f_5|) \tag{33E}$$

$$|\mathfrak{F}_\beta| + |\mathfrak{F}_\lambda| + |\mathfrak{F}_p| \le c(|F_2|_{-s} + |F_3|_{-s} + |f_5|) \tag{34E}$$

$$|\mathcal{M}| + |\mathcal{L}| + |\mathcal{P}| \le c(\varepsilon_0 + r_0)(|\psi^*|_{-s} + |\lambda^*|_{-s}) \tag{35E}$$

where $s > 0$ is an arbitrary number and c is independent on ε and r.

Proof. Let us consider the system of linear algebraic equations

$$\begin{aligned}
\boldsymbol{\Omega}\overline{\boldsymbol{\lambda}} + \overline{\mathbf{T}}\,\delta\boldsymbol{\beta} + p\,\mathbf{e}_1 &= \mathbf{a}, \\
\overline{\mathbf{T}}^\top\overline{\boldsymbol{\lambda}} + \overline{\mathbf{S}}\,\delta\boldsymbol{\beta} &= \mathbf{b}, \\
\overline{\lambda_1} &= f_5.
\end{aligned} \tag{36E}$$

Rewrite the first two equations in the form

$$\boldsymbol{\Omega}\overline{\boldsymbol{\lambda}} + \overline{\mathbf{T}_0}\,\delta\boldsymbol{\beta} + p\,\mathbf{e}_1 = \mathbf{a} + (\mathbf{T}_0 - \overline{\mathbf{T}})\delta\boldsymbol{\beta},$$

$$\mathbf{T}_0^\top\,\overline{\boldsymbol{\lambda}} + \mathbf{S}_0\,\delta\boldsymbol{\beta} = \mathbf{b} + (\mathbf{T}_0^\top - \overline{\mathbf{T}}^\top)\,\overline{\boldsymbol{\lambda}} + (\mathbf{S}_0 - \overline{\mathbf{S}})\,\delta\boldsymbol{\beta}.$$

Notice that

$$\overline{\boldsymbol{\lambda}} = \overline{\lambda}_2\mathbf{e}_2 + f_5\mathbf{e}_1,$$

where $\mathbf{e}_i$ are the basis vectors with the components $e_{ij} = \delta_{ij}$. Thus we get the linear system of the equations for $\overline{\lambda}_2$ and $\delta\boldsymbol{\beta}$

$$\overline{\lambda}_2 + \mathbf{t}_0 \cdot \delta\boldsymbol{\beta} = a_2 + \{(\mathbf{T}_0 - \overline{\mathbf{T}})\delta\boldsymbol{\beta}\}_2,$$

$$\mathbf{S}_0\delta\boldsymbol{\beta} + \lambda_2\,\mathbf{t}_0 = \mathbf{b} + f_5(\mathbf{T}_0^\top - \overline{\mathbf{T}}^\top)\mathbf{e}_1 + \tag{37E}$$

$$\overline{\lambda}_2(\mathbf{T}_0^\top - \overline{\mathbf{T}}^\top)\mathbf{e}_2 + (\mathbf{S}_0 - \overline{\mathbf{S}})\delta\boldsymbol{\beta}.$$

Express $\overline{\lambda}_2$ in terms of $\delta\boldsymbol{\beta}$ using the first equation in (37E). Substituting the result into the second equation in (37E) we obtain the following equation for $\delta\boldsymbol{\beta}$

$$\mathbf{K}_0\delta\boldsymbol{\beta} = (\mathbf{S}_0 - \overline{\mathbf{S}})\delta\boldsymbol{\beta} + \{(\mathbf{T}_0 - \overline{\mathbf{T}})\delta\boldsymbol{\beta}\}_2\big((\mathbf{T}_0^\top - \overline{\mathbf{T}}^\top)\boldsymbol{e}_2 - \mathbf{t}_0\big) - \delta\boldsymbol{\beta}\cdot\mathbf{t}_0\big((\mathbf{T}_0^\top - \overline{\mathbf{T}}^\top)\boldsymbol{e}_2$$
$$+\Big(\mathbf{b} - a_2\mathbf{t}_0 + a_2(\mathbf{T}_0^\top - \overline{\mathbf{T}}^\top)\boldsymbol{e}_2 + f_5(\mathbf{T}_0^\top - \overline{\mathbf{T}}^\top)\boldsymbol{e}_1\Big),$$

where the matrix $\mathbf{K}_0 = \mathbf{S}_0 - \mathbf{t}_0 \otimes \mathbf{t}_0$ has a bounded inverse. Thus we get the following equation for $\delta\boldsymbol{\beta}$

$$\delta\boldsymbol{\beta} - \mathbf{A}\delta\boldsymbol{\beta} = \mathbf{K}_0^{-1}(\mathbf{b} - a_2\mathbf{t}_0 + a_2(\mathbf{T}_0^\top - \overline{\mathbf{T}}^\top)\boldsymbol{e}_2 + f_5(\mathbf{T}_0^\top - \overline{\mathbf{T}}^\top)\boldsymbol{e}_1), \quad (38E)$$

where the linear mapping $\mathbf{A} : \mathbb{C}^2 \to \mathbb{C}^2$ is given by

$$\mathbf{A} : \delta\boldsymbol{\beta} \mapsto \mathbf{K}_0^{-1}\Big\{(\mathbf{S}_0 - \overline{\mathbf{S}})\delta\boldsymbol{\beta} + \{(\mathbf{T}_0 - \overline{\mathbf{T}})\delta\boldsymbol{\beta}\}_2\big((\mathbf{T}_0^\top - \overline{\mathbf{T}}^\top)\boldsymbol{e}_2 - \mathbf{t}_0\big) - \delta\boldsymbol{\beta}\cdot\mathbf{t}_0\big((\mathbf{T}_0^\top - \overline{\mathbf{T}}^\top)\boldsymbol{e}_2\Big\}.$$

In view of inequalities (3E), the mapping $\mathbf{A}$ admits the estimate $|\mathbf{A}\boldsymbol{\delta}\beta| \leq c(\varepsilon_0 + r_0)|\delta\boldsymbol{\beta}|$. Choosing ε_0 and r_0 so small that $c(\varepsilon_0 + r_0) \leq 1/2$, we obtain that equation (39E) has the only solution which admits the representation

$$\delta\boldsymbol{\beta} = (1 - \mathbf{A})^{-1}\mathbf{K}_0^{-1}\big(\mathbf{b} - a_2\mathbf{t}_0 + a_2(\mathbf{T}_0^\top - \overline{\mathbf{T}}^\top)\boldsymbol{e}_2 + f_5(\mathbf{T}_0^\top - \overline{\mathbf{T}}^\top)\boldsymbol{e}_1) \quad (39E)$$

When system (39E) is solved the vector $\overline{\boldsymbol{\lambda}}$ and the scalar p are restored by the relations

$$\overline{\boldsymbol{\lambda}} = f_5\mathbf{e}_1 + \{a_2 + ((\mathbf{T}_0 - \overline{\mathbf{T}})\boldsymbol{\beta}) \cdot \mathbf{e}_2\}\mathbf{e}_2 - \mathbf{t}_0 \cdot \delta\boldsymbol{\beta}\mathbf{e}_2 \qquad (40E)$$

$$p = a_1 + kf_5 + \{(\mathbf{T}_0 - \overline{\mathbf{T}})\boldsymbol{\beta}\}_1 \qquad (41E)$$

In view of (27E)-(28E) the vectors $\boldsymbol{\beta}$, $\boldsymbol{\lambda}$ and the constant p satisfy equations (36E) with the right hand sides

$$\mathbf{a} = \overline{F_2} - \overline{\mathbf{T}\nabla\psi_0}, \quad \mathbf{b} = \overline{F_3} - \overline{\mathbf{S}\nabla\psi_0} - \overline{\mathbf{T}^\top\boldsymbol{\lambda}^*}$$

Substituting these expressions into (39E)-(41E) we obtain desired representation (32Ea)-(32Ec) in which

$$\mathfrak{F}_\beta = (1 - \mathbf{A})^{-1}\mathbf{K}_0^{-1}\Big(\overline{F_3} - (\overline{F}_2 \cdot \mathbf{e}_2)\overline{\mathbf{t}}_0 + (\overline{F}_2 \cdot \mathbf{e}_2)(\mathbf{T}_0^\top - \overline{\mathbf{T}}^\top)\boldsymbol{e}_2 \qquad (42E)$$
$$+f_5(\mathbf{T}_0^\top - \overline{\mathbf{T}}^\top)\mathbf{e}_1\Big),$$

$$\mathfrak{F}_\lambda = f_5\mathbf{e}_1 + \overline{F}_2\mathbf{e}_2 + \Big((\mathbf{T}_0 - \overline{\mathbf{T}})\mathfrak{F}_\mu \cdot \mathbf{e}_2\Big)\mathbf{e}_2 - (\mathbf{t}_0 \cdot \mathfrak{F}_\mu)\mathbf{e}_2, \qquad (43E)$$

$$\mathfrak{F}_p = kf_5 + \overline{F}_2 + (\mathbf{T}_0 - \overline{\mathbf{T}})\mathfrak{F}_\mu \cdot \mathbf{e}_1 \qquad (44E)$$

$$, \qquad (45E)$$

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and

$$\mathcal{M} = (1 - \mathbf{A})^{-1}\mathbf{K}_0^{-1}\Big\{\big(\overline{\mathbf{T}\nabla\psi_0^*}\cdot\mathbf{e}_2\big)\mathbf{t}_0 - \big(\overline{\mathbf{T}\nabla\psi_0^*}\cdot\mathbf{e}_2\big)(\mathbf{T}_0^\top - \overline{\mathbf{T}}^\top)e_2 \quad (46\mathrm{E})$$
$$-\overline{\mathbf{S}\nabla\psi_0^*} - \overline{\mathbf{T}\boldsymbol{\lambda}^*}\Big\},$$
$$\mathcal{L} = -(\overline{\mathbf{T}\nabla\psi_0^*}\cdot\mathbf{e}_2)\mathbf{e}_2 - ((\mathbf{T}_0 - \overline{\mathbf{T}})\mathcal{M}\cdot\mathbf{e}_2)\mathbf{e}_2 - (\mathbf{t}_0\cdot\mathcal{M})\mathbf{e}_2, \quad (47\mathrm{E})$$
$$\mathcal{P} = -(\overline{\mathbf{T}\nabla\psi_0^*})\cdot\mathbf{e}_2 + (\mathbf{T}_0 - \overline{\mathbf{T}})\mathcal{M}\cdot\mathbf{e}_1. \quad (48\mathrm{E})$$

It remains to prove that these vectors and functionals satisfy inequalities (34E)-(35E). We begin with the observation that $\mathfrak{F}_\beta$, $\mathfrak{F}_\lambda$, and $\mathfrak{F}_p$ are linear integral functionals with analytic kernels of the linear space of functions F_2, F_3 and constants f_5. We have

$$|\mathfrak{F}_\beta| \le c(|\overline{F}_2| + |\overline{F}_3| + |f_5|),$$

which immediately gives (33E). In order to prove (34E) we note that the Cauchy inequality $|\overline{uv}| \le |u|_s|v|_{-s}$ implies

$$|\overline{F}_i| = |\overline{1\cdot F_i}| \le |1|_s\,|F_i|_{-s} \le c\,|F_i|_{-s},$$

gives (34E). Next notice that the $\mathcal{M}$, $\mathcal{L}$, and $\mathcal{P}$ are linear integral functionals with analytical kernels. In particular, they are continuous in every Banach space H_s. Hence estimate (35E) is almost trivial. We prove it for the functional $\mathcal{M}$. The same proof works for $\mathcal{L}$ and $\mathcal{P}$. In order to estimate $\mathcal{M}$, notice that

$$|\overline{\mathbf{T}\nabla\psi_0^*}| = |\overline{(\mathbf{T} - \mathbf{T}_0)\nabla\psi^*}| \le |\mathbf{T} - \mathbf{T}_0|_{s+1}|\nabla\psi^*|_{-s-1}$$
$$\le c\|\mathbf{T} - \mathbf{T}_0\|_{\sigma,0}|\psi^*|_{-s} \le cr_0|\psi^*|_{-s}.$$

Repeating these arguments we obtain

$$|\overline{\mathbf{S}\nabla\psi_0^*}| + |\overline{\mathbf{T}\boldsymbol{\lambda}^*}| \le cr_0\,(|\psi^*|_{-s} + |\boldsymbol{\lambda}^*|_{-s}).$$

Combining this result withy the obvious inequality

$$|\mathcal{M}| \le c(|\overline{\mathbf{T}\nabla\psi_0^*}| + |\overline{\mathbf{S}\nabla\psi_0^*}| + |\overline{\mathbf{T}\boldsymbol{\lambda}^*}|)$$

we obtain estimate (35E) for $\mathcal{M}$. $\qquad\square$

Let us turn to the proof of Proposition E.5. Our strategy is the following. First, we use representations (32Ea)-(32Ec) in order to reduce the system (26E) (31E) to the closed system of equations for the deviations ψ_0^*, $\boldsymbol{\lambda}^*$, and $\boldsymbol{\chi}^*$. Then we reduce the obtained system to an abstract operator equation. We solve this equation in the space of bounded functions by using contraction mapping principle. Finally we prove that the obtained solution is analytic. Let us consider the basic system of equations (29E)-(31E). Assume that ε_0 and r_0 meet all requirements of Lemma E.6. It follows from representations (32Ea)-(32Ec) in this lemma that we can rewrite equations (29E)-(31E) in the equivalent form

$$\boldsymbol{\partial}\psi_0^* = G_1^*,$$
$$\mathbf{J}\boldsymbol{\partial}\boldsymbol{\lambda}^* + \boldsymbol{\Omega}\boldsymbol{\lambda}^* + (\mathbf{T}\nabla\psi_0^*)^* = G_2^* - \mathbf{T}^*\mathcal{M}(\psi_0^*, \boldsymbol{\lambda}^*), \qquad (49E)$$
$$-\boldsymbol{\partial}\boldsymbol{\chi}^* + (\mathbf{S}\nabla\psi_0^*)^* + (\mathbf{T}^\top\boldsymbol{\lambda}^*) = G_3^* - \mathbf{S}^*\mathcal{M}(\psi_0^*, \boldsymbol{\lambda}^*) - (\mathbf{T}^\top)^*\mathcal{L}(\psi_0^*, \boldsymbol{\lambda}^*),$$

where

$$G_1^* = F_1^*, \quad G_2^* = F_2^* - \mathbf{T}^*\mathfrak{F}_\beta, \quad G_3^* = F_3^* - \mathbf{S}^*\mathfrak{F}_\beta - (\mathbf{T}^\top)^*\mathfrak{F}_\lambda$$

It is worth noting that equations (49E) and relations (32Ea)-(32Ec) forms the system of equations which is equivalent to equations (23Ea)-(23Ed) It follows from estimate (34E) that

$$\|G_1^*\|_{\sigma_1,0} + \|G_2^*\|_{\sigma_1,0} + \|G_3^*\|_{\sigma_1,0} \leq c(\|F_1^*\|_{\sigma_1,0} + \|F_2^*\|_{\sigma_1,0} + \|F_3^*\|_{\sigma_1,0} + |f_5|).$$
$$(50E)$$

Now introduce the linear operator

$$\Xi_1 : (F_1^*, F_2^*, F_3^*) \to (\psi_0^*, \boldsymbol{\lambda}^*, \boldsymbol{\chi}^*)$$

which assigns to every vector (F_1^*, F_2^*, F_3^*) the solution of the following system of equations

$$\boldsymbol{\partial}\psi_0^* = F_1^*, \qquad (51E)$$
$$\mathbf{J}\boldsymbol{\partial}\boldsymbol{\lambda}^* + \boldsymbol{\Omega}\boldsymbol{\lambda}^* + (\mathbf{T}\nabla\psi_0^*)^* = F_2^*, \qquad (52E)$$
$$-\boldsymbol{\partial}\boldsymbol{\chi}^* + (\mathbf{S}\nabla\psi_0^*)^* + (\mathbf{T}^\top\boldsymbol{\lambda}^*) = F_3^*. \qquad (53E)$$

Denote by $\mathcal{Z}_\sigma$ the subspace of the Banach space $\mathcal{A}_{\sigma_0} \times \mathcal{A}_{\sigma_0}^2 \times \mathcal{A}_{\sigma,0}^{n-1}$ which consists of all functions with with zero mean value. Let us show that for every $0 \leq \sigma_0 < \sigma_1 \leq \sigma$, the operator $\Xi_1 : \mathcal{Z}_{\sigma_1} \to \mathcal{Z}_{\sigma_0}$ is continuous and

$$\|\Xi_1(F_1^*, F_2^*, F_3^*)\|_{Z_{\sigma_0}} \leq c(\sigma_1 - \sigma_0)^{-4n-6}\|\Xi_1(F_1^*, F_2^*, F_3^*)\|_{Z_{\sigma_1}} \qquad (54E)$$

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Choose an arbitrary $(F_1^*, F_2^*, F_3^*) \in \mathcal{Z}_{\sigma_1}$. Since the system (51E)-(53E) is triangular, the existence of solution to this system obviously follows from Lemmas E.2 and E.3. Next, estimate (7E) in Lemma E.2 implies

$$\|\psi_0^*\|_{2/3\sigma_0 + 1/3\sigma_1, 0} \leq c(\sigma_1 - \sigma_0)^{-n-2} \|F_1^*\|_{\sigma_1, 0}. \tag{55E}$$

Using this inequality and applying Lemma E.3 to equation (52E) we arrive at the estimate

$$\begin{aligned}
\|\boldsymbol{\lambda}^*\|_{2/3\sigma_0 + 1/3\sigma_1, 0} &\leq c(\sigma_1 - \sigma_0)^{-2n-3} \big(\|\psi_0^*\|_{1/3\sigma_0 + 2/3\sigma_1, 0} + \|F_2^*\|_{1/3\sigma_0 + 2/3\sigma_1, 0} \big) \\
&\leq c(\sigma_1 - \sigma_0)^{-3n-5} \big(\|F_1^*\|_{\sigma_1, 0} + \|F_2^*\|_{\sigma_1, 0} \big).
\end{aligned} \tag{56E}$$

Finally applying Lemma E.2 to equation (53E) we obtain

$$\begin{aligned}
\|\boldsymbol{\chi}^*\|_{\sigma_0, 0} &\leq c(\sigma_1 - \sigma_0)^{-n-2} \big(\|\psi_0^*\|_{2/3\sigma_0 + 1/3\sigma_1, 0} + \|\boldsymbol{\lambda}^*\|_{2/3\sigma_0 + 1/3\sigma_1, 0} + \|F_3^*\|_{2/3\sigma_0 + 1/3\sigma_1, 0} \big) \\
&\leq c(\sigma_1 - \sigma_0)^{-4n-6} \big(\|F_1^*\|_{\sigma_1, 0} + \|F_2^*\|_{\sigma_1, 0} + \|F_3\|_{\sigma_1, 0} \big).
\end{aligned} \tag{57E}$$

Combining estimates (55E)-(57E) we obtain (54E). Now introduce the second linear operator

$$\Xi_2 : (\psi_0^*, \boldsymbol{\lambda}^*, \boldsymbol{\chi}^*) \to \Big(0, \mathbf{T}^* \mathcal{M}(\psi_0^*, \boldsymbol{\lambda}^*), \mathbf{S}^* \mathcal{M}(\psi_0^*, \boldsymbol{\lambda}^*) + (\mathbf{T}^\top)^* \mathcal{L}(\psi_0^*, \boldsymbol{\lambda}^*) \Big)^\top$$

Since the embedding $\mathcal{A}_{0,0} \to H_{-s}$ is bounded for $s \geq 0$, estimate (35E) yields the inequality

$$\|\Xi_2(\psi_0^*, \boldsymbol{\lambda}^*, \boldsymbol{\chi}^*)\|_{\mathcal{Z}_\sigma} \leq c(\varepsilon_0 + r_0) \|(\psi_0^*, \boldsymbol{\lambda}^*, \boldsymbol{\chi}^*)|_{\mathcal{Z}_0}. \tag{58E}$$

Now we can rewrite system (49E) in the form of the operator equation

$$(\psi_0^*, \boldsymbol{\lambda}^*, \boldsymbol{\chi}^*) = \Xi_1 \Xi_2 (\psi_0^*, \boldsymbol{\lambda}^*, \boldsymbol{\chi}^*) + \Xi_1 (G_1^*, G_2^*, G_3^*). \tag{59E}$$

Estimates (54E), (58E), and the inequality $1/4 < \sigma$ imply

$$\|\Xi_1 \Xi_2 (\psi_0^*, \boldsymbol{\lambda}^*, \boldsymbol{\chi}^*)\|_{\mathcal{Z}_0} \leq c(\varepsilon_0 + r_0) \|(\psi_0^*, \boldsymbol{\lambda}^*, \boldsymbol{\chi}^*)\|_{\mathcal{Z}_0} \tag{60E}$$

It follows that the norm of the operator $\Xi_1 \Xi_2 : \mathcal{Z}_0 \to \mathcal{Z}_0$ does not exceed $c(\varepsilon_0 + r_0)$. Choosing ε_0 and r_0 sufficiently small and applying the contraction mapping principle we conclude that operator equation (59E) has a unique solution in the space $\mathcal{Z}_0$. This solution satisfies the inequality

$$\|(\psi_0^*, \boldsymbol{\lambda}^*, \boldsymbol{\chi}^*)\|_{\mathcal{Z}_0} \leq c \|\Xi_1 (G_1^*, G_2^*, G_3^*)\|_{\mathcal{Z}_0}. \tag{61E}$$

Let us prove that the obtained solution is analytic. Since $1/4 \leq \sigma_1$, estimates (54E) with $\sigma_0 = 0$ and estimate (50E) imply the inequality

$$\|\Xi_1(G_1^*, G_2^*, G_3^*)\|_{\mathcal{Z}_0} \leq \|(G_1^*, G_2^*, G_3^*)\|_{\mathcal{Z}_{\sigma_1}}$$
$$\leq c(\|F_1^*\|_{\sigma_1,0} + \|F_2^*\|_{\sigma_1,0} + \|F_3^*\|_{\sigma_1,0} + |f_5|). \quad (62\mathrm{E})$$

Hence the solution to equation (59E) admits the estimate

$$\|(\psi_0^*, \boldsymbol{\lambda}^*, \boldsymbol{\chi}^*)\|_{\mathcal{Z}_0} \leq c(\|F_1^*\|_{\sigma_1,0} + \|F_2^*\|_{\sigma_1,0} + \|F_3^*\|_{\sigma_1,0} + |f_5|),$$

which along with (58E) leads to the inequality

$$\|\Xi_2(\psi_0^*, \boldsymbol{\lambda}^*, \boldsymbol{\chi}^*)\|_{\mathcal{Z}_\sigma} \leq c(\|F_1^*\|_{\sigma_1,0} + \|F_2^*\|_{\sigma_1,0} + \|F_3^*\|_{\sigma_1,0} + |f_5|). \quad (63\mathrm{E})$$

On the other hand, inequality (54E) yields

$$\left\|\Xi_1\Big(\Xi_2(\psi_0^*, \boldsymbol{\lambda}^*, \boldsymbol{\chi}^*) + (G_1^*, G_2^*, G_3^*)\Big)\right\|_{\mathcal{Z}_{\sigma_0}}$$
$$\leq c(\sigma_1 - \sigma_0)^{-4n-6}\Big(\|\Xi_2(\psi_0^*, \boldsymbol{\lambda}^*, \boldsymbol{\chi}^*)\|_{\mathcal{Z}_{\sigma_1}} + \|(G_1^*, G_2^*, G_3^*)\|_{\mathcal{Z}_{\sigma_1}}\Big)$$

Combining this result with (63E) and (50E) we finally obtain

$$\left\|\Xi_1\Big(\Xi_2(\psi_0^*, \boldsymbol{\lambda}^*, \boldsymbol{\chi}^*) + (G_1^*, G_2^*, G_3^*)\Big)\right\|_{\mathcal{Z}_{\sigma_0}}$$
$$\leq c(\sigma_1 - \sigma_0)^{-4n-6}(\|F_1^*\|_{\sigma_1,0} + \|F_2^*\|_{\sigma_1,0} + \|F_3^*\|_{\sigma_1,0} + |f_5|)$$

Using this result and equation (59E) we finally obtain that the solution to system (49E) admits the estimate

$$\|\psi_0^*\|_{\sigma_0,0} + \|\boldsymbol{\lambda}^*\|_{\sigma_0,0} + \|\boldsymbol{\chi}^*\|_{\sigma_0,0}$$
$$\leq c(\sigma_1 - \sigma_0)^{-4n-6}c(|f_5| + \|F_1\|_{\sigma_1,0} + \|F_2\|_{\sigma_1,0} + \|F_3\|_{\sigma_1,0}) \quad (64\mathrm{E})$$

Hence we proof that for all sufficiently small ε_0 and r_0 system (49E) has an analytic solution satisfying inequality (64E). Next notice that the mean value $\overline{\boldsymbol{\chi}} = f_3$ and the constant $q = f_1$. Recall that the vector $\delta\boldsymbol{\beta}$, the mean value $\boldsymbol{\lambda}^*$, and the constant p are connected with the deviations $\nabla\psi^*$, $\boldsymbol{\lambda}^*$ and $\boldsymbol{\chi}^*$ by relations (32Ea)-(32Ec). It follows from this relations and estimates (34E)-(35E) that

$$|\overline{\boldsymbol{\chi}}| + |\delta\boldsymbol{\beta}| + |\overline{\boldsymbol{\lambda}}| + |p| + |q| \leq c(|f_1| + |f_3| + |f_5| + \|F_1\|_{\sigma_1,0} + \|F_2\|_{\sigma_1,0} + \|F_3\|_{\sigma_1,0})$$
$$+ c(\|\psi_0^*\|_{\sigma_0,0} + \|\boldsymbol{\lambda}^*\|_{\sigma_0,0} + \|\boldsymbol{\chi}^*\|_{\sigma_0,0})$$

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Combining this result with (64E) we obtain

$$|\overline{\boldsymbol{\chi}}| + |\delta\boldsymbol{\beta}| + |\overline{\boldsymbol{\lambda}}| + |p| + |q|$$
$$\leq c(\sigma_1 - \sigma_0)^{-4n-6}(|f_1| + |f_3| + |f_5| + \|F_1\|_{\sigma_1,0} + \|F_2\|_{\sigma_1,0} + \|F_3\|_{\sigma_1,0}).$$

This result, estimate (64E), and decomposition (25E) imply desired estimate (24E). This completes the proof of Proposition E.5.

$\square$

Now we consider the extended truncated system, which includes the extra equation for the matrix-valued function $\boldsymbol{\Gamma}$. This extended system is formulated as follows

$$\boldsymbol{\partial}\psi_0 + q = F_1, \tag{65Ea}$$

$$\mathbf{J}\boldsymbol{\partial}\boldsymbol{\lambda} + \mathbf{T}\boldsymbol{\mu} + p\mathbf{e}_1 = F_2, \quad \boldsymbol{\mu} = \delta\boldsymbol{\beta} + \nabla\psi_0 \tag{65Eb}$$

$$-\boldsymbol{\partial}\boldsymbol{\chi} + \mathbf{S}\boldsymbol{\mu} + \mathbf{T}^\top\boldsymbol{\lambda} = F_3, \tag{65Ec}$$

$$\overline{\psi_0} = f_1, \quad \overline{\boldsymbol{\chi}} = f_3 \quad \overline{\lambda_1} = f_5, \tag{65Ed}$$

$$\boldsymbol{\partial}(\mathbf{J}\boldsymbol{\Gamma}) + \boldsymbol{\Omega}\boldsymbol{\Gamma} + (\boldsymbol{\Omega}\boldsymbol{\Gamma})^\top + \tag{65Ee}$$

$$\mathbf{U}_{ij}\frac{\partial\lambda_i}{\partial\xi_j} + \mu_i\mathbf{E}_i + \lambda_i\mathbf{K}_i + \delta\mathbf{M} = F_4,$$

$$\Gamma_{11} = -\Gamma_{22}, \quad \overline{\Gamma}_{12} = f_4, \quad \delta\mathbf{M} = \delta M \text{ diag } (1,0). \tag{65Ef}$$

The following proposition guarantees the well-posedness of the truncated problem (65E).

Proposition E.7. *Under the assumptions of Theorem E.1, there are $\varepsilon_0 > 0$ and r_0 with the following properties. For every*

$$(\alpha, k) \in \Sigma_\varrho, \quad \|\boldsymbol{\varphi} - \boldsymbol{\varphi}(\alpha)\|_{\sigma,d} \leq r_0, \quad \boldsymbol{\varphi}(\alpha) = (0,0,0,\boldsymbol{\alpha},1,0,0), \quad |\varepsilon| \leq \varepsilon_0, \quad ,$$

$$0 \leq \sigma_0 < \sigma_1 < \sigma, \quad \sigma_1 > 1/4$$

$$\mathbf{f} \in \mathbb{C}^4, \quad \mathbf{F} \in \mathcal{F}_{\sigma_1,0}.$$

Then problem (65E) *has a unique solution*

$$(\psi_0, \boldsymbol{\lambda}, \boldsymbol{\chi}, \boldsymbol{\Gamma}, \delta\boldsymbol{\beta}, q, p, \delta\mathbf{M}) \in \mathcal{A}_{\sigma_0,0} \times \mathcal{A}_{\sigma_0,0}^2 \times \mathcal{A}_{\sigma,d}^{n-1} \times \mathcal{A}_{\sigma_0,0}^4 \times \mathbb{C}^{n-1} \times \mathbb{C}^3.$$

This solution admits the estimate

$$\|(\psi_0, \boldsymbol{\lambda}, \boldsymbol{\chi}, \boldsymbol{\Gamma})\|_{\sigma_0,0} + |(\delta\boldsymbol{\beta}, \delta q, \delta p, \delta\mathbf{M})| \leq c(\sigma_1 - \sigma_0)^{-8n-12}(\|\mathbf{F}\|_{\sigma_1,0} + |\mathbf{f}|)$$
(66E)

where the constant c is independent of ε_0, r_0, and σ.

Proof. We begin with the observation that equations (65Ea)-(65Ed) are independent of Γ. Moreover, system (65Ea)-(65Ed) coincides with system (23Ea)-(23Ed). Applying Proposition E.5 we conclude that this system has a unique analytic solution. This solution admits the estimate

$$\|\psi_0\|_{\sigma_0,0} + \|\boldsymbol{\lambda}\|_{\sigma_0,0} + \|\boldsymbol{\chi}\|_{\sigma_0,0} + |p| + |q|$$
$$+ |\delta\boldsymbol{\beta}| \leq c(\sigma_1 - \sigma_0)^{-6n-8}(\|\mathbf{F}\|_{\sigma_1,0} + |\mathbf{f}|). \quad (67E)$$

Hence it suffices to analyze equation (65Ee). Rewrite it in the form

$$\boldsymbol{\partial}(\mathbf{J}\boldsymbol{\Gamma}) + \boldsymbol{\Omega}\boldsymbol{\Gamma} + (\boldsymbol{\Omega}\boldsymbol{\Gamma})^{\top} + \delta\mathbf{M} = F_4 - \mathbf{U}_{ij}\frac{\partial\lambda_i}{\partial\xi_j} - \mu_i\mathbf{E}_i - \lambda_i\mathbf{K}_i \qquad (68E)$$

Applying inequality (67E) we obtain

$$\|\mathbf{U}_{ij}\frac{\partial\lambda_i}{\partial\xi_j} - \mu_i\mathbf{E}_i - \lambda_i\mathbf{K}_i\|_{2/3\sigma_0+1/3\sigma_1,0} \leq c(\sigma_1 - \sigma_0)^{-1}(\|\boldsymbol{\lambda}\|_{1/2\sigma_0+1/2\sigma_1,0}$$
$$+ c\|\psi_0\|_{1/2\sigma_0+1/2\sigma_1,0} + |\delta\boldsymbol{\beta}|) \leq c(\sigma_1 - \sigma_0)^{-6n-9}(\|\mathbf{F}\|_{\sigma_1,0} + |\mathbf{f}|). \qquad (69E)$$

Applying Lemma 17E we conclude that equation (68E) has a unique analytic solution which satisfies condition (65Ef) and admits the estimate

$$\|\boldsymbol{\Gamma}\|_{\sigma_0,0} + |\delta\mathbf{M}| \leq c(\sigma_1 - \sigma_0)^{-2n-3} \left(\|\mathbf{F}_4 - \mathbf{U}_{ij}\frac{\partial\lambda_i}{\partial\xi_j} - \mu_i\mathbf{E}_i - \lambda_i\mathbf{K}_i\|_{2/3\sigma_0+1/3\sigma_1,0} + |f_4| \right)$$

It follows from this and (69E) that

$$\|\boldsymbol{\Gamma}\|_{\sigma_0,0} + |\delta\mathbf{M}| \leq c(\sigma_1 - \sigma_0)^{-8n-12}(\|\mathbf{F}\|_{\sigma_1,0} + |\mathbf{f}|).$$

Combining this result with (67E) we obtain (66E) and the proposition follows.

□

Proof of Theorem E.1   We are now in a position to complete the proof of Theorem E.1. The proof is based on the Proposition E.7 and the contraction mapping principle. By technical reasons it is convenient to introduce the following denotations. Introduce the vectors

$$\mathfrak{w} = (\psi_0, \boldsymbol{\lambda}, \boldsymbol{\chi}, \boldsymbol{\Gamma}, \delta\boldsymbol{\beta}, q, p, \delta M), \quad \mathfrak{f} = (F_1, F_2, F_3, F_4, f_1, f_3, f_4, f_5).$$

Denote by $\mathcal{X}_\sigma$ the closed subspace of the Banach space

$$\mathcal{A}_{\sigma_0,0} \times \mathcal{A}_{\sigma_0,0}^2 \times \mathcal{A}_{\sigma,d}^{n-1} \times \mathcal{A}_{\sigma_0,0}^4 \times \mathbb{C}^{n-1} \times \mathbb{C}^3,$$

which consists of all vector functions $\mathfrak{w}$ such that $\operatorname{tr}\boldsymbol{\Gamma} = 0$. Denote by $\mathcal{Y}_\sigma$ the closed subspace of the Banach space

$$\mathcal{A}_{\sigma_0,0} \times \mathcal{A}_{\sigma_0,0}^2 \times \mathcal{A}_{\sigma,d}^{n-1} \times \mathcal{A}_{\sigma_0,0}^4 \times \mathbb{C} \times \mathbb{C}^{n-1} \times \mathbb{C}^{n-1} \times \mathbb{C}$$

which consists of all vectors $\mathfrak{f}$ such that the matrix F_4 is symmetric. Rewrite main system (1E) in the form.

$$\boldsymbol{\partial}\psi_0 + q = F_1 + \Xi_{3,1}(p, \delta M), \tag{70Ea}$$

$$\mathbf{J}\boldsymbol{\partial}\boldsymbol{\lambda} + \boldsymbol{\Omega}\boldsymbol{\lambda} + \mathbf{T}\boldsymbol{\mu} + p\mathbf{e}_1 = F_2 + \Xi_{3,2}(p, \delta M), \quad \boldsymbol{\mu} = \nabla\psi_0 + \delta\boldsymbol{\beta} \tag{70Eb}$$

$$-\boldsymbol{\partial}\boldsymbol{\chi} + \mathbf{S}\boldsymbol{\mu} + \mathbf{T}^\top\boldsymbol{\lambda} = F_3, \tag{70Ec}$$

$$\boldsymbol{\partial}(\mathbf{J}\boldsymbol{\Gamma}) + \boldsymbol{\Omega}\boldsymbol{\Gamma} + (\boldsymbol{\Omega}\boldsymbol{\Gamma})^\top + \mathbf{U}_{ij}\frac{\partial\lambda_i}{\partial\xi_j} + \mu_i\mathbf{E}_i + \lambda_i\mathbf{K}_i + \delta\mathbf{M} = \tag{70Ed}$$

$$F_4 + \Xi_{3,4}(\delta M), \quad \Gamma_{11} = -\Gamma_{22},$$

$$\overline{\psi_0} = f_1 + \Xi_{3,5}(\boldsymbol{\beta}, \boldsymbol{\lambda}) \tag{70Ee}$$

$$\overline{\boldsymbol{\chi}} = f_3 + \Xi_{3,6}(\boldsymbol{\chi}), \tag{70Ef}$$

$$\overline{\Gamma_{12}} = f_4 + \Xi_{3,7}(\boldsymbol{\Gamma}) \tag{70Eg}$$

$$\overline{\lambda_1} = f_5 + \Xi_{3,8}(\boldsymbol{\lambda}, \boldsymbol{\chi}), \tag{70Eh}$$

where

$$\Xi_{3,1}(p, \delta M) = -p(w_1 - \alpha) - \frac{1}{2}\delta M(w_1 - \alpha)^2,$$

$$\Xi_{3,2}(p, \delta M) = -p(\mathbf{W}^\top - \mathbf{I})\mathbf{e}_1 - \delta M(w_1 - \alpha)\mathbf{W}^\top\mathbf{e}_1,$$

$$\Xi_{3,3} = 0, \quad \Xi_{3,4}(\delta M) = \delta\mathbf{M} - \mathbf{W}^\top\delta\mathbf{M}\mathbf{W},$$

$$\Xi_{3,5}(\boldsymbol{\beta}, \boldsymbol{\lambda}) = \boldsymbol{\delta\beta} \cdot \overline{\mathbf{u}} - \overline{w_2\mathbf{W}\boldsymbol{\lambda} \cdot \mathbf{e}_1}$$

$$\Xi_{3,6}(\boldsymbol{\chi}) = \overline{(\mathbf{I} - \mathbf{V}^{-\top})\boldsymbol{\chi}}, \quad \Xi_{3,7}(\boldsymbol{\Gamma}) = \overline{\left((\mathbf{I} - \mathbf{W})\boldsymbol{\Gamma}\right)}_{12}$$

$$\Xi_{3,8}(\boldsymbol{\lambda}, \boldsymbol{\chi}) = \overline{(\mathbf{I} - \mathbf{W})\boldsymbol{\lambda} \cdot \mathbf{e}_1} - \overline{\boldsymbol{\chi} \cdot \nabla w_1}.$$

It is necessary to prove that, under the assumptions of Theorem E.1, the system of equations (70E) for all $\mathfrak{f} \in \mathcal{Y}_{\sigma_1}$ has a unique solution $\mathfrak{w} \in \mathcal{X}_{\sigma_0}$ satisfying the inequality

$$\|\mathfrak{w}\|_{\mathcal{X}_{\sigma_0}} \leq c(\sigma_1 - \sigma_0)^{-8n-12} \|\mathfrak{f}\|_{\mathcal{Y}_{\sigma_1}} \tag{71E}$$

Recall that

$$0 \leq \sigma_0 < \sigma_1 \leq \sigma, \quad 1/4 \leq \sigma_1. \tag{72E}$$

The proof is based on the following auxiliary lemma.

Lemma E.8. *Under the assumptions of Theorem E.1 the operator* $\Xi_3 = \left(\Xi_{3,i}\right)_{1\leq i \leq 8}$ *admits the estimate*

$$\|\Xi_3(\mathfrak{w})\|_{\mathcal{Y}_\sigma} \leq cr_0\|\mathfrak{w}\|_{\mathcal{X}_0}. \tag{73E}$$

Proof. Obviously we have

$$\|\Xi_{3,1}(p, \delta M)\|_{\mathcal{Y}_\sigma} + \|\Xi_{3,2}(p, \delta M)\|_{\mathcal{Y}_\sigma} + \|\Xi_{3,4}(\delta M)\|_{\mathcal{Y}_\sigma} \leq$$
$$c(|p| + |\delta M|)(\|w_1 - \alpha\|_{\sigma,0} + \|\mathbf{W} - \mathbf{I}\|_{\sigma,0})$$

It follows from the estimate

$$(\alpha, k) \in \Sigma_\varrho, \quad \|\boldsymbol{\varphi} - \boldsymbol{\varphi}(\alpha)\|_{\sigma,d} \leq r, \quad \boldsymbol{\varphi}(\alpha) = (0, 0, 0, \boldsymbol{\alpha}, 1, 0, 0),$$

in condition of Theorem E.1 and the relations $\boldsymbol{\varphi} = (\boldsymbol{\beta}, \varphi_0, \mathbf{u}, \mathbf{w}, W_{11}, W_{12}, W_{21})$, $\det \mathbf{W} = 1$ that

$$\|\mathbf{u}\|_{\sigma,0} + \|w_2\|_{\sigma,0} + \|w_1 - \alpha\|_{\sigma,0} + \|\mathbf{W} - \mathbf{I}\|_{\sigma,0} \leq cr \leq cr_0. \tag{74E}$$

Thus we get

$$\|\Xi_{3,1}(p, \delta M)\|_{\mathcal{Y}_\sigma} + \|\Xi_{3,2}(p, \delta M)\|_{\mathcal{Y}_\sigma} + \|\Xi_{3,4}(\delta M)\|_{\mathcal{Y}_\sigma} \leq cr_0(|p| + |\delta M|) \tag{75E}$$

Next, the Cauchy inequality implies

$$|\Xi_{3,5}| + |\Xi_{3,6}| + |\Xi_{3,7}| + |\Xi_{3,8}| \leq c(|\delta\boldsymbol{\beta}| + |\boldsymbol{\lambda}|_0 + |\boldsymbol{\chi}|_0 + |\boldsymbol{\Gamma}|_0) \times$$
$$(|\mathbf{u}|_0 + |\mathbf{W} - \mathbf{I}|_0 + |w_2\mathbf{W}|_0 + |\nabla w_1|_0 + |\mathbf{I} - \mathbf{V}^{-1}|_0)$$

Recall that $I - \mathbf{V}^{-\top} \equiv -\mathbf{u}'$ and the embedding

$$\mathcal{A}_{\sigma,0} \hookrightarrow H_1 \hookrightarrow H_0, \quad \mathcal{A}_{0,0} \hookrightarrow H_0$$

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is bounded. From this and (74E) we obtain

$$|\mathbf{u}'|_0 + |\mathbf{W} - \mathbf{I}|_0 + |w_2\mathbf{W}|_0 + |\nabla w_1|_0 + |\mathbf{I} - \mathbf{V}^{-\top}|_0$$
$$\leq |\mathbf{u}|_1 + |\mathbf{W} - \mathbf{I}|_0 + |w_2\mathbf{W}|_0 + |w_1^*|_1$$
$$\leq \|\mathbf{u}\|_{\sigma,0} + \|\mathbf{W} - \mathbf{I}\|_{\sigma,0} + \|w_2\mathbf{W}\|_{\sigma,0} + \|w_1^*\|_{\sigma,0} \leq cr_0$$

and
$$|\boldsymbol{\lambda}|_0 + |\boldsymbol{\chi}|_0 + |\boldsymbol{\Gamma}|_0 \leq c(\|\boldsymbol{\lambda}\|_{0,0} + \|\boldsymbol{\chi}\|_{0,0} + \|\boldsymbol{\Gamma}\|_{0,0}).$$

Combining the obtained results we arrive at the estimate

$$|\Xi_{3,5}| + |\Xi_{3,6}| + |\Xi_{3,7}| + |\Xi_{3,8}| \leq cr_0(|\boldsymbol{\beta}| + \|\boldsymbol{\lambda}\|_{0,0} + \|\boldsymbol{\chi}\|_{0,0} + \|\boldsymbol{\Gamma}\|_{0,0}). \quad (76E)$$

It remains to note that the desired estimate (73E) obviously follows from (75E) and (76E). $\qquad\square$

Let us turn to the proof of the Theorem E.1. Denote by Ξ_4 the linear operator which assigns to every $\mathfrak{f} \in Y_{\sigma_1}$ the solution $\mathfrak{w}$ of problem (65E). It follows from the Proposition (E.7) that for small ε_0 and r_0, the operator $\Xi_4 : \mathcal{Y}_{\sigma_1} \to \mathcal{X}_{\sigma_0}$ is bounded and

$$\|\Xi_4\mathfrak{f}\|_{\sigma_0} \leq c(\sigma_1 - \sigma_0)^{-6n-8}\|\mathfrak{f}\|_{\mathcal{Y}_{\sigma_1}}. \quad (77E)$$

In particular, we have

$$\|\Xi_4\mathfrak{f}\|_{\sigma_0} \leq c\|\mathfrak{f}\|_{\mathcal{Y}_{\sigma_1}}. \quad (78E)$$

Now we can rewrite system (70E) in the form of the operator equation

$$\mathfrak{w} = \Xi_4\left(\Xi_3\,\mathfrak{w} + \mathfrak{f}\right). \quad (79E)$$

By virtue of (73E) and (78E) the operator $\Xi_4\Xi_3 : \mathcal{X}_0 \to \mathcal{X}_0$ is bounded and its norm does not exceed cr_0. Choosing r_0 sufficiently small and applying the contraction mapping principle we conclude that operator equation (79E) has a unique solution in the space $\mathcal{X}_0$. In view of (78E), this solution admits the estimate

$$\|\mathfrak{w}\|_{\mathcal{X}_0} \leq c\,\|\mathfrak{f}\|_{Y_{\sigma_1}}.$$

Let us prove that this solution is analytic. Notice that estimate (73E) implies

$$\|\Xi_3\mathfrak{w}\|_{\mathcal{Y}_{\sigma_1}} \leq c\|\mathfrak{w}\|_{\mathcal{X}_0} \leq c\|\mathfrak{f}\|_{Y_{\sigma_1}}$$

Hence
$$\|\Xi_3\mathfrak{w} + \mathfrak{f}\|_{\mathcal{Y}_{\sigma_1}} \leq c\|\mathfrak{w}\|_{\mathcal{X}_0} + \|\mathfrak{f}\|_{\mathcal{Y}_{\sigma_1}} \leq c\|\mathfrak{f}\|_{Y_{\sigma_1}}.$$

Combining this result with (77E) we obtain desired estimate (71E). This completes the proof of Theorem E.1.

E.2 Proof of Theorem 8.1

We split the proof into two parts. First we prove the existence and uniqueness of solutions to problem (8.5),

$$\boldsymbol{\partial}\,\boldsymbol{\mu} = -p\nabla w_1 - \nabla g_1, \tag{80Ea}$$

$$\mathbf{J}\boldsymbol{\partial}\,\boldsymbol{\lambda} + \boldsymbol{\Omega}\,\boldsymbol{\lambda} + \mathbf{T}\,\boldsymbol{\mu} + p\mathbf{W}^\top\mathbf{e}_1 = \mathbf{g}_2 \tag{80Eb}$$

$$-\boldsymbol{\partial}\,\boldsymbol{\lambda} + \mathbf{S}\boldsymbol{\mu} + \mathbf{T}^\top\boldsymbol{\lambda} = 0 \tag{80Ec}$$

$$\overline{\{W\boldsymbol{\lambda}\}\cdot\mathbf{e}_1} + \overline{\{\boldsymbol{\chi}\cdot\nabla w_1\}} = \gamma, \tag{80Ed}$$

$$\overline{\{\mathbf{V}^{-\top}\boldsymbol{\chi}\}} = 0. \tag{80Ee}$$

in the space of analytic function and establish estimate (8.9). Next, we estimate the obtained solution in the Sobolev space H_s. We begin with the observation that problem (80E) is the very particular case of general problem (1E). Notice that, in view of conditions (8.6) and (8.7) of Theorem 8.1, the matrices $\mathbf{W}$, $\mathbf{V}$, $\mathbf{S}$, $\mathbf{T}$ and the function w_1 meet all requirements of Theorem E.1 with σ replaced by $\sigma/2$ and $r_0 = c\varepsilon_0$. Next set

$$\mathbf{U}_{ij} = 0, \quad \mathbf{E}_i = 0, \quad \mathbf{K}_i = 0, \tag{81E}$$

$$\sigma_0 = \sigma/4, \quad \sigma_1 = \sigma/2 \geq 1/4$$

Obviously these quantities satisfy all conditions of Theorem E.1. Now introduce the temporary notation

$$F_1 = -g_1^*, \quad F_2 = \mathbf{g}_2,, \quad F3 = 0, \quad F_4 = 0,$$
$$f_1 = 0, \quad f_3 = 0, \quad f_4 = 0, \quad f_5 = \gamma, \tag{82E}$$

Since $r_0 \leq c\varepsilon_0$, it follows from Theorem E.1 that for a suitable choice of ε_0 and for all $|\varepsilon| \leq \varepsilon_0$, the equations (1E) with the righthand sides (82E) have a unique analytic solution. This solution satisfies inequalities (5E). In particular, we have

$$|(\psi_0, \boldsymbol{\lambda}, \boldsymbol{\chi})\|_{3/8\sigma,0} + |(\delta\boldsymbol{\beta}, p)| \leq c(\|g_1^*\|_{\sigma/2,0} + \|\mathbf{g}_2^*\|_{\sigma/2,0} + |\gamma|). \tag{83E}$$

By virtue of condition (81E) and the equalities $f_4 = 0$, $F_4 = 0$, the matrices $\boldsymbol{\Gamma}$ and $\delta\mathbf{M}$ satisfy the homogeneous equation

$$\boldsymbol{\partial}(\mathbf{J}\boldsymbol{\Gamma}) + \boldsymbol{\Omega}\boldsymbol{\Gamma} + (\boldsymbol{\Omega}\boldsymbol{\Gamma})^\top + \delta\mathbf{M} = \delta\mathbf{M} - \mathbf{W}^\top\delta\mathbf{M}\mathbf{W},$$

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$$\overline{\Gamma_{12}} = \overline{\left((\mathbf{I} - \mathbf{W})\mathbf{\Gamma}\right)}_{12}$$

Applying Lemma E.3 and the contraction mapping principle we conclude that $\mathbf{\Gamma} = \delta\mathbf{M} = 0$. Since $\overline{F}_1 = 0$, we also have $q = 0$. Hence the functions ψ_0, $\boldsymbol{\lambda}$ and $\boldsymbol{\chi}$ satisfy the equations

$$\boldsymbol{\partial}\,\psi_0 \;=\; -pw_1^* - g_1^*, \tag{84Ea}$$

$$\mathbf{J}\boldsymbol{\partial}\,\boldsymbol{\lambda} \,+\, \boldsymbol{\Omega}\,\boldsymbol{\lambda} \,+\, \mathbf{T}\,\boldsymbol{\mu} + p\mathbf{W}^\top\mathbf{e}_1 = \mathbf{g}_2 \tag{84Eb}$$

$$\boldsymbol{\mu} = \delta\boldsymbol{\beta} + \nabla\psi_0,$$

$$-\boldsymbol{\partial}\,\boldsymbol{\lambda} \,+\, \mathbf{S}\boldsymbol{\mu} \,+\, \mathbf{T}^\top\boldsymbol{\lambda} = 0 \tag{84Ec}$$

$$\overline{\{W\boldsymbol{\lambda}\}\cdot\mathbf{e}_1} \;+\; \overline{\{\boldsymbol{\chi}\cdot\nabla w_1\}} = \gamma, \tag{84Ed}$$

$$\overline{\{\mathbf{V}^{-\top}\boldsymbol{\chi}\}} \;=\; 0. \tag{84Ee}$$

It follows that $\boldsymbol{\mu}, \boldsymbol{\lambda}, \boldsymbol{\chi}$ and p satisfy equations (80E). Inequalities (83E) and the obvious inequality

$$\|\boldsymbol{\mu}\|_{\sigma/4,0} \le c(|\boldsymbol{\beta}| + \|\psi_0\|_{3/8\sigma,0}$$

implies the estimate

$$|(\boldsymbol{\mu},\boldsymbol{\lambda},\boldsymbol{\chi})\|_{\sigma/4,0} + |p| \le c(\|g_1^*\|_{\sigma/2,0} + \|\mathbf{g}_2^*\|_{\sigma/2,0} + |\gamma|). \tag{85E}$$

which yields (8.9). It remains to estimate the solution to problem (80E) in the Sobolev spaces. Now we change the denotations and set

$$F_1 = -pw_1^* - g_1*, \quad F_2 = p(\mathbf{I} - \mathbf{W}^\top)\mathbf{e}_1 + \boldsymbol{g}_2, \quad F_3 = 0,$$
$$f_1 = 0, \quad f_3 = \overline{(\mathbf{I} - \mathbf{V}^{-\top})\boldsymbol{\chi}} \quad f_5 = \overline{(\mathbf{I} - \mathbf{W})\boldsymbol{\lambda}\mathbf{e}_1} - \overline{\boldsymbol{\chi}\cdot\nabla w_1} \tag{86E}$$

Now we estimate these quantities in the Sobolev space H_s. Obviously we have

$$|F_1|_s \le |p||w_1^*|_s + |g_1^*|_s \le c\varepsilon_0|p| + |g_1^*|_s,$$
$$|F_2|_s \le |p||\mathbf{I} - \mathbf{W}|_s + |\mathbf{g}_2|_s \le c\varepsilon_0|p| + |\mathbf{g}_2|_s. \tag{87E}$$

We also have

$$|\overline{F}_2| \le c\varepsilon_0|p| + |\overline{\mathbf{g}_2}| \tag{88E}$$

The Cauchy inequality implies

$$|f_3| \leq |\mathbf{I} - \mathbf{V}^{-\top}|_{-s} \, |\boldsymbol{\chi}|_s \leq \|\mathbf{I} - \mathbf{V}^{-\top}\|_{\sigma/2,0} \, |\boldsymbol{\chi}|_s \leq c\varepsilon_0 |\boldsymbol{\chi}|_s.$$

$$|f_5| \leq |\mathbf{I} - \mathbf{W}|_{-s} \, |\boldsymbol{\lambda}|_s + |\nabla w_1|_{-s} |\boldsymbol{\chi}|_s \leq$$
$$c\|\mathbf{I} - \mathbf{W}\|_{\sigma/2,0} |\boldsymbol{\lambda}|_s + \|w_1^*\|_{\sigma/2,0} |\boldsymbol{\chi}|_s \leq c\varepsilon_0 (|\boldsymbol{\lambda}|_s + |\boldsymbol{\chi}|_s)$$

Combining the obtained results we arrive at the inequality

$$|f_3| + |f_5| \leq c\varepsilon_0 (|\overline{\boldsymbol{\lambda}}| + |\overline{\boldsymbol{\chi}}|) + c\varepsilon_0 (|\boldsymbol{\lambda}^*|_s + |\boldsymbol{\chi}^*|_s) \tag{89E}$$

With this notation equations (84E) can be written in the form

$$\boldsymbol{\partial}\psi_0 + q = F_1, \tag{90Ea}$$
$$\mathbf{J}\boldsymbol{\partial}\boldsymbol{\lambda} + \boldsymbol{\Omega}\boldsymbol{\lambda} + \mathbf{T}\boldsymbol{\mu} + p \, \mathbf{e}_1 = F_2, \quad \boldsymbol{\mu} = \delta\boldsymbol{\beta} + \nabla\psi_0 \tag{90Eb}$$
$$-\boldsymbol{\partial}\boldsymbol{\chi} + \mathbf{S}\boldsymbol{\mu} + \mathbf{T}^\top \boldsymbol{\lambda} = F_3, \tag{90Ec}$$

$$\overline{\psi_0} = f_1, \quad \overline{\boldsymbol{\chi}} = f_3, \quad \overline{\lambda_1} = f_5. \tag{90Ed}$$

Hence the these equations coincide with equations (23E) at least formally. It follows from Lemma E.6 that a solution to system (80E) admits the representation (32E). recalling identities $\boldsymbol{\beta} = \overline{\boldsymbol{\mu}}$ and $\nabla\psi_0 = \boldsymbol{\mu}^*$ we can rewrite this representation in the form

$$\overline{\boldsymbol{\mu}} = \mathcal{M}(\boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) + \mathfrak{F}_\beta, \tag{91Ea}$$
$$\overline{\boldsymbol{\lambda}} = \mathcal{L}(\boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) + \mathfrak{F}_\lambda, \tag{91Eb}$$
$$p = \mathcal{P}(\boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) + \mathfrak{F}_p. \tag{91Ec}$$

It follows from estimates (34E) and (35E) in Lemma E.6 that the constant vectors $\mathfrak{F}_\beta$, $\mathfrak{F}_\lambda$, $\mathfrak{F}_p$ and the linear functionals $\mathcal{M}$, $\mathcal{L}$, $\mathcal{P}$ admit the estimates

$$|\mathfrak{F}_\beta| + |\mathfrak{F}_\lambda| + |\mathfrak{F}_p| \leq c(|F_2|_s + |F_3|_s + |f_5|), \tag{92E}$$

$$|\mathcal{M}| + |\mathcal{L}| + |\mathcal{P}| \leq c\varepsilon_0 (|\boldsymbol{\mu}^*|_s + |\boldsymbol{\lambda}^*|_s). \tag{93E}$$

Moreover, estimate (33E) yields the inequality

$$|\overline{\mathfrak{F}_\beta}| \leq c|\overline{F}_2|. \tag{94E}$$

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Combining estimates (87E), (89E) and (92E), (93E) we arrive at the inequality

$$|\mathfrak{F}_\beta| + |\mathfrak{F}_\lambda| + |\mathfrak{F}_p| + |\mathcal{M}| + |\mathcal{L}| + |\mathcal{P}| \le c\varepsilon_0(|\overline{\boldsymbol{\lambda}}| + |\overline{\boldsymbol{\chi}}| + |p|) +$$
$$+ c\varepsilon_0(|\boldsymbol{\lambda}^*|_s + |\boldsymbol{\chi}^*|_s) + c(|g_1^*|_s + |\mathbf{g}_2|_s).$$

¿From this and (91E) we obtain

$$|\overline{\boldsymbol{\lambda}}| + |\overline{\boldsymbol{\chi}}| + |\overline{\boldsymbol{\mu}}| + |p| \le c\varepsilon_0(|\overline{\boldsymbol{\lambda}}| + |\overline{\boldsymbol{\chi}}| + |p|) +$$
$$+ c\varepsilon_0(|\boldsymbol{\lambda}^*|_s + |\boldsymbol{\chi}^*|_s) + c(|g_1^*|_s + |\mathbf{g}_2|_s).$$

Choosing ε_0 sufficiently small we finally obtain

$$|\overline{\boldsymbol{\lambda}}| + |\overline{\boldsymbol{\chi}}| + |\overline{\boldsymbol{\mu}}| + |p| \le c\varepsilon_0(|\boldsymbol{\lambda}^*|_s + |\boldsymbol{\chi}^*|_s) + c(|g_1^*|_s + |\mathbf{g}_2|_s). \qquad (95E)$$

Next, equations (80Ea)-(80Ec) imply the equalities

$$\boldsymbol{\partial}\,\boldsymbol{\mu}^* = -p\nabla w_1 - \nabla g_1,$$
$$\boldsymbol{\partial}\,\boldsymbol{\lambda}^* + \boldsymbol{\Omega}\,\boldsymbol{\lambda}^* = \mathbf{g}_2 - \mathbf{T}\,\boldsymbol{\mu}^* - (\mathbf{T}\,\overline{\boldsymbol{\mu}} + p\mathbf{W}^\top\mathbf{e}_1)$$
$$\boldsymbol{\partial}\,\boldsymbol{\chi}^* = (\mathbf{S}\boldsymbol{\mu}^* + \mathbf{T}^\top\boldsymbol{\lambda}^*) + (\mathbf{S}\overline{\boldsymbol{\mu}} + \mathbf{T}^\top\overline{\boldsymbol{\lambda}})$$

Applying Lemma E.2 to the first equation and noting that the embedding of every Sobolev space into $\mathcal{A}_{\sigma/2,0}$ is bounded we obtain

$$|\boldsymbol{\mu}^*|_{s+3n+4} \le c(|p| + |g_1^*|_{s+4n+6})$$

Next, applying Lemma E.3 to the second equation we obtain

$$|\boldsymbol{\lambda}^*|_{s+n+1} \le c|\mathbf{g}_2|_{s+3n+4} + c|\boldsymbol{\mu}^*|_{s+3n+4} + c(|p| + |\overline{\boldsymbol{\mu}}|).$$

Applying Lemma E.2 to the third equation we obtain

$$|\boldsymbol{\chi}^*|_s \le c(|\boldsymbol{\mu}^*|_{s+n+1} + |\boldsymbol{\lambda}^*|_{s+n+1}) + c(|p| + |\overline{\boldsymbol{\mu}} + |\overline{\boldsymbol{\lambda}}|)$$

Combining the obtained results we arrive at the inequality

$$|\boldsymbol{\mu}^*|_s + |\boldsymbol{\lambda}^*|_s + |\boldsymbol{\chi}^*|_s \le c(|g_1^*|_{s+4n+6} + |\mathbf{g}_2|_{s+4n+6}) + c(|p| + |\overline{\boldsymbol{\mu}} + |\overline{\boldsymbol{\lambda}}|). \quad (96E)$$

Combining (95E) and (96E), and choosing ε_0 sufficiently small we obtain the desired estimate

$$|\boldsymbol{\mu}|_s + |\boldsymbol{\lambda}|_s + |\boldsymbol{\chi}|_s + |p| \le c(|g_1^*|_{s+4n+6} + |\mathbf{g}_2|_{s+4n+6}), \qquad (97E)$$

which gives estimate (8.10) in Theorem 8.1. Next, estimates (88E) and (94E) yields

$$|\overline{\mathfrak{F}_\beta}| \le c\varepsilon_0|p| + |\overline{\mathbf{g}_2}| \tag{98E}$$

which along with (91Ea) and (93E) implies

$$|\overline{\boldsymbol{\mu}}| \le |\overline{\mathbf{g}_2}| + c\varepsilon_0(|p| + |\boldsymbol{\mu}^*|_s + |\boldsymbol{\lambda}^*|_s)$$

Combining this result withy (97E), we finally obtain

$$|\overline{\boldsymbol{\mu}}| \le |\overline{\mathbf{g}_2}| + c\varepsilon_0(|g_1^*|_{s+4n+6} + |\mathbf{g}_2|_{s+4n+6}),$$

which gives estimate (8.11) in Theorem 8.1. This completes the proof of Theorem 8.1.

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